Lecture 9: LQG in I/O form

- Problem formulation
- Background – State-space form
- Heuristic solution
- Examples
- Computational procedure
- Interpretation
- Summary

LQG in state space

Process

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) + v(k) \\ y(k) &= Cx(k) + e(k) \end{aligned}$$

Filter

$$\begin{aligned} \hat{x}(k+1|k) &= \Phi \hat{x}(k|k-1) + \Gamma u(k) + K\varepsilon(k) \\ \varepsilon(k) &= y(k) - C\hat{x}(k|k-1) \end{aligned}$$

Controller

$$u(k) = -L\hat{x}(k|k-1) - M\varepsilon(k)$$

Optimal system dynamics $P(z) = \det(zI - \Phi + \Gamma L)$

Optimal filter dynamics $C(z) = \det(zI - \Phi + KC)$

How to obtain P and C?

Optimal system dynamics

Assume stationarity

$$P(z) = \det(zI - \Phi + \Gamma L)$$

For the SISO case ($Q_1 = C^T C$ and $Q_2 = \rho$)

$$\rho A(z^{-1})A(z) + B(z^{-1})B(z) = rP(z^{-1})P(z)$$

where $r = \Gamma^T S \Gamma + \rho$. Will be proved later!

Spectral factorization

- $P(z)$ monic $\deg P = \deg A = n$
- $P(z)$ roots inside or on the unit circle
  Inside if $\rho \neq 0$
- $\rho \to 0 \quad P(z) = z^d B(z)/b_0$
  Possibly reflected in the unit circle

Optimal filter dynamics

State space representation for

$$\begin{aligned} y(k) &= \frac{B(q)}{A(q)} u(k) + \frac{C(q)}{A(q)} e(k) \\ \frac{C(q)}{A(q)} &= \frac{q^n + c_1 q^{n-1} + \dots c_n}{q^n + a_1 q^{n-1} + \dots a_n} = \frac{C(q) - A(q)}{A(q)} + 1 \end{aligned}$$

on observer form

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) + \Gamma_v e(k) \\ y(k) &= Cx(k) + e(k) \end{aligned}$$

$$\begin{aligned} C &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \\ \Gamma^T &= \begin{pmatrix} 0 & \dots & b_0 & \dots & b_{n-d} \end{pmatrix} \\ \Gamma_v^T &= \begin{pmatrix} c_1 - a_1 & c_2 - a_2 & \dots & c_n - a_n \end{pmatrix} \end{aligned}$$

Optimal filter dynamics, cont'd

The Kalman filter is given by

$$\hat{x}(k+1) = (\Phi - KC)\hat{x}(k) + \Gamma u(k) + Ky(k)$$

with $K = \Gamma_v$!

Thus

$$\det(qI - (\Phi - KC)) = C(q)$$

Roots of $C(z)$ inside the unit circle

Heuristic solution

Process ($\deg A = \deg C = n$)

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

No common factors between A and B

Controller

$$u(k) = -\frac{S(q)}{R(q)}y(k) \qquad \deg R = n$$

Closed loop system (Diophantine equation)

$$A(z)R(z) + B(z)S(z) = P(z)C(z)$$

Solution?
Uniqueness??

Solution and uniqueness

Diophantine equation $\deg P = \deg C = n$

$$A(z)R(z) + B(z)S(z) = P(z)C(z)$$

Case 1 – Delay in the controller

$\deg S = n - 1$ and $\deg R = n$
$\Rightarrow$ exists unique solution (Theorem 5.1)
$2n$ coefficients and $2n$ equations

Case 2 – No delay in the controller

$\deg S = n$ and $\deg R = n$
$2n + 1$ coefficients and $2n$ equations
$\Rightarrow$ Extra condition required. How to get that?

Case 2 – No delay in the controller

Lemma 12.2: $P(z)$ given by spectral factorization, $A(z)$ be monic, $A(z)$ and $B(z)$ no common roots outside the unit disc or on the unit circle; then there exists a unique solution to the equations

$$A^*(z)X(z) + rP(z)S^*(z) = B(z)C^*(z)$$
$$z^d B^*(z)X(z) - rP(z)R^*(z) = -\rho A(z)C^*(z)$$

with $\deg X(z) < n$, $\deg R^*(z) \leq n$ and $\deg S^*(z) < n$, where $n = \deg A(z)$.

$$S(z) = z^n S^*(z^{-1}), \qquad R(z) = z^n R^*(z^{-1}), \qquad S(0) = 0$$

The two identities can be written as

$$P^*(z)X(z) = B(z)R^*(z) - \rho A(z)S^*(z)$$

LQG I/O case (Theorem 12.4)

Assume:

1. $\deg A(z) = \deg C(z) = n$
2. All the zeros of $C(z)$ inside the unit disc
3. No factors common to $A(z)$, $B(z)$, $C(z)$
4. A possible common factor of $A(z)$ and $B(z)$ has all its zeros inside the unit disc.

Let the monic polynomial $P(z)$ have all its zeros inside the unit disc and $\deg P(z) = n$. The optimal control law with no delay is

$$u(k) = -\frac{S^*(q^{-1})}{R^*(q^{-1})}\, y(k) = -\frac{S(q)}{R(q)}\, y(k)$$

where $R^*(z)$ and $S^*(z)$ are the unique solution to (Lemma 12.2) with $\deg X(z) < n$.

LQG I/O case,cont'd

The resulting closed loop system is

$$y(k) = \frac{R(q)}{P(q)}\, e(k), \qquad u(k) = -\frac{S(q)}{P(q)}\, e(k)$$

and the minimal value of the loss function is

$$\min \mathrm{E}(y^2 + \rho u^2) = \frac{\sigma^2}{2\pi i} \oint \frac{R(z)R(z^{-1}) + \rho S(z)S(z^{-1})}{P(z)P(z^{-1})} \frac{dz}{z}$$

Sketch of the proof

Transform the control variable

$$u = v - \frac{S}{R}\, y$$

v is a transformed control variable to be determined

$$y = \frac{BRv + CRe}{AR + BS} = \frac{BRv + CRe}{PC} = \frac{BR}{PC}\, v + \frac{R}{P}\, e$$

Then

$$u = v - \frac{SBv + SCe}{PC} = \frac{PC - BS}{PC}\, v - \frac{S}{P}\, e = \frac{AR}{PC}\, v - \frac{S}{P}\, e$$

Sketch of the proof, cont'd

The loss function can be written as

$$\begin{aligned} J &= \mathrm{E}(y^2 + \rho u^2) = \mathrm{E}\left(\frac{BR}{PC}\, v + \frac{R}{P}\, e\right)^2 + \rho \mathrm{E}\left(\frac{AR}{PC}\, v - \frac{S}{P}\, e\right)^2 \\ &= J_1 + 2J_2 + J_3 \end{aligned}$$

J_1 depends only on v^2, J_3 only on e^2, and J_2 contains cross terms.

For causal controllers with no time delay $v(t) = V(q)e(t)$, where $V(q)$ is a rational function with zero pole excess.

$$J_2 = \frac{\sigma^2}{2\pi i} \oint \frac{B(z)R(z)R(z^{-1}) - \rho A(z)R(z)S(z^{-1})}{P(z)C(z)P(z^{-1})}\, V(z) \frac{dz}{z}$$

Sketch of the proof, cont'd

But

$$B(z)R(z^{-1}) - \rho A(z)S(z^{-1}) = P(z^{-1})X(z)$$

Hence

$$J_2 = \frac{\sigma^2}{2\pi i}\oint \frac{R(z)X(z)}{P(z)C(z)}\, V(z)\,\frac{dz}{z} = \mathrm{E}\left(\Big(\frac{R(q)X(q)}{P(q)C(q)}\, v(k)\Big) e(k)\right)$$

$P(z)$ and $C(z)$ are stable implies that $\deg X(z) < n$ and

$$\deg R(z)X(z) < \deg P(z)C(z) = 2n$$

The quantity

$$\frac{R(q)X(q)}{P(q)C(q)}\, v(k)$$

is thus a function of $e(k-1), e(k-2), \ldots$ independent of $e(k) \quad \Rightarrow \quad J_2 = 0 \quad \Rightarrow \quad J_1$ minimum for $v = 0$

Under the carpet

What about:

- Common factors between the polynomials A, B, and C.
- The proof that $S(0) = 0$ is the condition to use.
- How the optimal controller is derived directly.
- The case $A(0) = 0$.
- Zeros on the unit circle

Example – Unstable zero

$$A(z) = (z-1)(z-0.7)$$
$$B(z) = 0.9z + 1$$
$$C(z) = z(z-0.7)$$

Spectral factorization

$$rP(z)P(z^{-1}) = \rho A(z)A(z^{-1}) + B(z)B(z^{-1})$$

Diophantine equation

$$P(z)C(z) = A(z)R(z) + B(z)S(z)$$

gives here

$$R(z) = z(z + r_1) \qquad S(z) = s_0 z(z-0.7)$$
$$r_1 = \frac{1 + p_1 - 0.9p_2}{1.9} \qquad s_0 = \frac{1 + p_1 + p_2}{1.9}$$

How to find P?

Spectral factorization

Solve

$$rP(z)P(z^{-1}) = \rho A(z)A(z^{-1}) + B(z)B(z^{-1})$$

or the algebraic Riccati equation. Messy calculations. Use Matlab

Polynomial identity

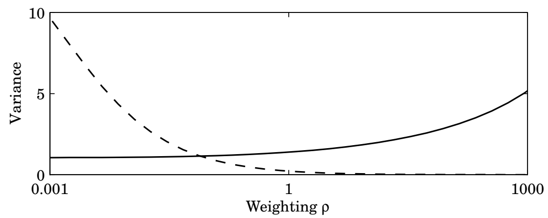
```
rho=1;
Astar=A(end:-1:1);
Bstar=Ba(end:-1:1);
rhs=rho*conv(Astar,A)...
+conv(Bstar,Ba);
pp=dsort(roots(rhs));
P=poly(pp(end-2+1:end));
```

Algebraic Riccati eq.

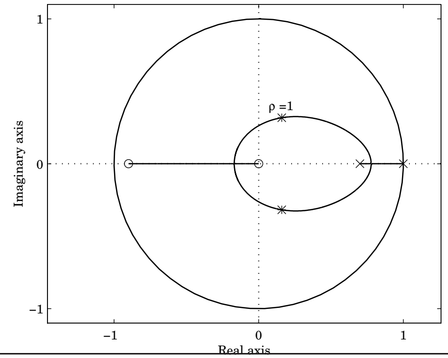
```
Phi=[1.7 1;-0.7 0];
Gam=[0.9; 1];
Q1=[1 0;0 0];
[L,S,E]=dlqr(Phi,Gam,Q1,rho);
P=poly(E);
```

Variation of ρ

Input (dashed) and output (full) variances

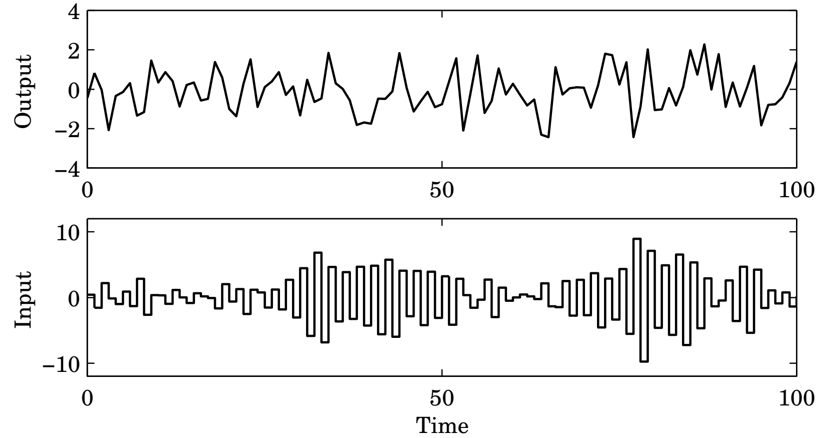


Closed loop poles



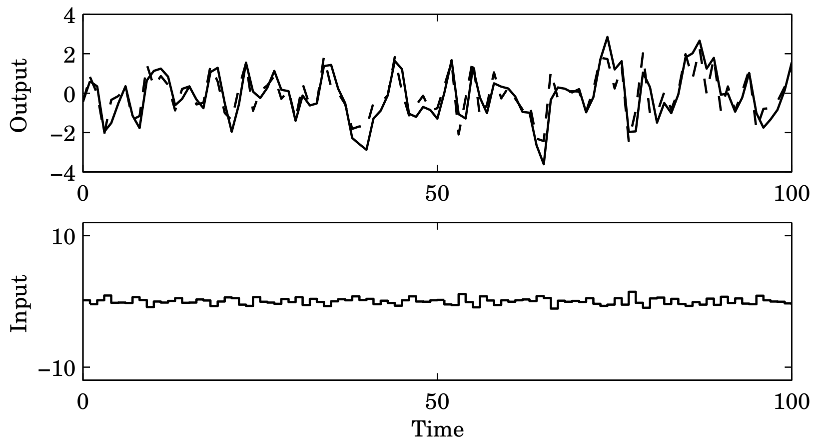
Simulation

Output and input when $\rho = 0$



Simulation

Output when $\rho = 0$ (dashed) and $\rho = 1$ (full). Input when $\rho = 1$



Computational procedure

1. Rewrite into standard ABC-form
2. Spectral factorization to get $P(z)$.
   Possibly stable common factors of A and B will appear in P

3a. If $A(0) \neq 0$ solve Diophantine equation with $\deg R = \deg S = n$ and $S(0) = 0$
    If non-uniqueness go to Step 3b

3b. If $A(0) = 0$ solve Diophantine equation and

$$A(z)R(z) + B(z)S(z) = P(z)C(z)$$
$$P^*(z)X(z) + \rho A(z)S^*(z) = R^*(z)B(z)$$

Unstable common factors?

Interpretation and extensions

- Close connection with state-space formulation and pole placement
- The optimization gives a unique solution
- Uncontrollable and unstable modes requires new loss function. Useful to introduce integral action.

Gain margin for discrete-time LQ

The Riccati equation can be written as (11.37)

$$\rho + \frac{B(z^{-1})B(z)}{A(z^{-1})A(z)} = r(1 + H_1(z^{-1}))(1 + H_1(z))$$

where

$$H_1(z) = L(zI - \Phi)^{-1}\Gamma$$

The return difference

$$1 + H_1(z) = 1 + L(zI - \Phi)^{-1}\Gamma = \frac{P(z)}{A(z)}$$

Thus

$$H_1(z) = \frac{P(z) - A(z)}{A(z)}$$

Gain margin for discrete-time LQ cont'd

Use the controller

$$u(k) = -\beta L x(k)$$

The return difference is now

$$1 + \beta H_1(z)$$

The stability of the closed-loop system is determined from

$$A(z) + \beta\Big(P(z) - A(z)\Big) = 0$$

Finite gain margin seen from root locus arguments

Compare the continuous-time case

$$0.5 < \beta < \infty$$

Summary

- Input-output formulation, innovation model

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

- Diophantine equation
- Prediction
- Minimum variance control, stable and unstable inverse
- LQG – Uniqueness through optimization
- Everything is connected