Lecture 8:
Optimal design: I/O methods

- Relation between state-space and I/O formulations
- Shortcuts using polynomials
- Problem formulation
- Optimal prediction
- Connection to the Kalman filter
- Minimum varaiance control

LQG in state space

Process

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) + v(kh) \\ y(kh) &= Cx(kh) + e(kh) \end{aligned}$$

Controller

$$\begin{aligned} \hat{x}(k+1|k) &= \Phi \hat{x}(k|k-1) + \Gamma u(k) + K(k)\varepsilon(k) \\ \varepsilon(k) &= y(k) - C\hat{x}(k|k-1) \\ u(k) &= -L\hat{x}(k|k-1) - M\varepsilon(k) \end{aligned}$$

where

$$M = \begin{cases} 0 & u(k|Y_{k-1}) \\ LK_f + L_v K_v & u(k|Y_k) \end{cases}$$

L and K obtained through Riccatiequations

Optimal system and filter dynamics

Optimal system dynamics: $(\Phi - \Gamma L)$
Optimal filter dynamics: $(\Phi - KC)$
in both cases $u(k|Y_{k-1})$ and $u(k|Y_k)$!

Best seen using statevariables x, $\tilde{x}$

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}(k+1) = \begin{bmatrix} \Phi - \Gamma L & \Gamma(L - MC) \\ 0 & \Phi - KC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}(k) + \begin{bmatrix} I \\ I \end{bmatrix} v(k) + \begin{bmatrix} -\Gamma M \\ -K \end{bmatrix} e(k)$$

Optimal system dynamics $\Phi - \Gamma L$

Closed loop characteristic polynomial

$$P(z) = \det(zI - \Phi + \Gamma L)$$

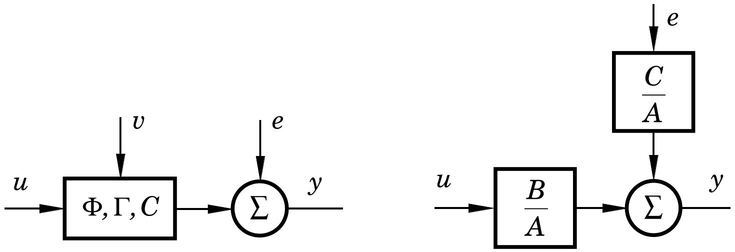
for the SISO loss function case
($Q_1 = C^T C$ and $Q_2 = \rho$)
can be calculated from (Theorem 11.4)

$$\rho A(z^{-1})A(z) + B(z^{-1})B(z) = rP(z^{-1})P(z)$$

Compare with spectral factorization!

Derived from stationary Riccati equation.
Actually $r = \Gamma^T S \Gamma + \rho$.

Process model



Two noise sources.
Reduce to one (for $R_{12} = 0$)

$$y = \frac{B(q)}{A(q)}u + \frac{B_v(q)}{A(q)}v + e$$

Introduce equivalent noise

Process model – Equivalent noise

$$Ay = Bu + B_v v + Ae$$

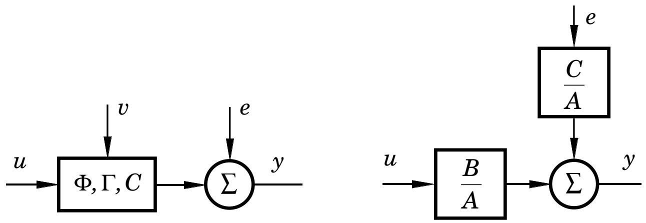
Use spectral factorization

$$\sigma_v^2 B_v(z)B_v(z^{-1}) + \sigma_e^2 A(z)A(z^{-1}) = \sigma_\varepsilon^2 C(z)C(z^{-1})$$

where $C(z)$ stable.

Innovations representation

$$A(q)y(k) = B(q)u(k) + C(q)\varepsilon(k)$$



Optimal filter dynamics $\Phi - KC$

Innovations representation for

$$\begin{aligned} y(k) &= \frac{q^n + c_1 q^{n-1} + \dots c_n}{q^n + a_1 q^{n-1} + \dots a_n} e(k) = \frac{C(q)}{A(q)} e(k) \\ &= \frac{C(q) - A(q)}{A(q)} e(k) + e(k) \end{aligned}$$

on observable form

$$\begin{aligned} x(k+1) &= \Phi x(k) + K e(k) \\ y(k) &= C x(k) + e(k) \\ C &= [1 \quad 0 \quad \dots \quad 0] \\ K^T &= [c_1 - a_1 \quad c_2 - a_2 \quad \dots \quad c_n - a_n] \end{aligned}$$

Optimal filter dynamics $\Phi - KC$

Kalman filter

$$\hat{x}(k+1) = (\Phi - KC)\hat{x}(k) + Ky(k)$$

The characteristic polynomial is

$$\det(zI - (\Phi - KC)) = C(z)$$

where $C(z)$ stable

Controller dynamics

$$q\hat{x} = (\Phi - KC)\hat{x} + \Gamma u + Ky$$
$$u = -(L - MC)\hat{x} - My$$

Solve for $\hat{x}$

$$\hat{x} = [qI - (\Phi - KC)]^{-1} (\Gamma u + Ky)$$

Controller

$$u = -(L - MC)\,(qI - (\Phi - KC))^{-1}\,(\Gamma u + Ky) - My$$
$$= -\frac{Q(q)}{C(q)}u - \frac{S(q)}{C(q)}y \quad \text{where} \quad C(z) = \det(zI - (\Phi - KC)$$

Polynomial form of the controller

$$\underbrace{(C(q) + Q(q))}_{R(q)}\, u(k) = -S(q)y(k)$$

Problem formulation

Process model

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

$$\begin{aligned} \deg A(z) &= n & A(z) &= z^n + a_1 z^{n-1} + \cdots + a_n \\ \deg C(z) &= n & C(z) &= z^n + c_1 z^{n-1} + \cdots + c_n \\ \deg B(z) &= n - d & B(z) &= b_0 z^{n-d} + \cdots + b_{n-d} \end{aligned}$$

$C(z)$ stable, otherwise equivalent noise $C^+(z)C^{-*}(z)$

Criterion

$$J_{lq} = \mathrm{E}[y^2(k) + \rho u^2(k)]$$

Admissible control laws

$$u(k) = f(y(k), y(k-1), \ldots, u(k-1), \ldots)$$

Shortcuts using polynomials? For $\rho = 0$?

Summing up

Process

$$y(k) = \frac{B(q)}{A(q)}u(k) + \frac{C(q)}{A(q)}e(k)$$

Loss function

$$\mathrm{E}\left\{ \sum_{k=0}^{\infty} (y^2(k) + \rho u^2(k)) \right\}$$

Controller

$$u(k) = -\frac{S(q)}{R(q)}y(k)$$

THEN closed loop characteristic polynomial

$$A(z)R(z) + B(z)S(z) = P(z)C(z)$$

where

$$P(z) = \det(zI - (\Phi - \Gamma L)) \qquad C(z) = \det(zI - (\Phi - KC))$$

Prediction – Heuristic derivation

Determine $\hat{y}(k+m|k)$ for

$$y(k) = \frac{C(q)}{A(q)}\,e(k) = \frac{C^*(q^{-1})}{A^*(q^{-1})}\,e(k)$$

$$y(k+m) = \frac{C(q)}{A(q)}e(k+m) = \sum_{i=0}^{\infty} f_i e(k+m-i)$$
$$= \underbrace{\left\{ e(k+m) + f_1 e(k+m-1) + \cdots + f_{m-1}e(k+1) \right\}}_{\tilde{y}(k+m|k) \text{ Unknown at time k}}$$
$$+ \underbrace{\left\{ f_m e(k) + f_{m+1}e(k-1) + \cdots \right\}}_{\hat{y}(k+m|k), \text{ Computable at time k}}$$

Use

$$e(k) = \frac{A(q)}{C(q)}y(k) \qquad \deg A(q) = \deg C(q) = n, \quad C(q) \text{ stable}$$

Prediction – Formal solution

Introduce the identity

$$q^{m-1}C(q) = A(q)F(q) + G(q)$$

or as quotient and remainder, $\deg G(q) < n$

$$q^{m-1}\frac{C(q)}{A(q)} = F(q) + \frac{G(q)}{A(q)}$$

Thus

$$\begin{aligned} y(k+m) &= \frac{C(q)}{A(q)}q^m e(k) = F(q)e(k+1) + \frac{qG(q)}{A(q)}e(k) \\ &= F(q)e(k+1) + \frac{qG(q)}{C(q)}y(k) \end{aligned}$$

Predictor and prediction error

$$\hat{y}(k+m|k) = \frac{qG(q)}{C(q)}y(k)$$

$$\tilde{y}(k+m|k) = y(k+m) - \hat{y}(k+m|k) = F(q)e(k+1)$$

Variance of prediction error

$$\mathrm{E}\tilde{y}(k+m|k)^2 = \left(1 + f_1^2 + \cdots + f_{m-1}^2\right)\sigma^2$$

Interpretation of predictor

$$\hat{y}(k+m|k) = \frac{qG(q)}{C(q)}y(k)$$

$$\tilde{y}(k+m|k) = F(q)e(k+1)$$

$$\mathrm{E}\tilde{y}(k+m|k)^2 = \left(1 + f_1^2 + \cdots + f_{m-1}^2\right)\sigma^2$$

- Linear predictor
- Stable predictor dynamics
- $\tilde{y}(k+1|k) = e(k+1)$ Innovation
- Same as the stationary Kalman filter

- What happens with increasing prediction horizon?

Example

$$A(q) = q^2 - 1.5q + 0.7 \qquad C(q) = q^2 - 0.2q + 0.5$$

3 step ahead prediction

$$q^2(q^2 - 0.2q + 0.5) = (q^2 - 1.5q + 0.7)(q^2 + f_1q + f_2) + g_0q + g_1$$

Triangular linear system of equations

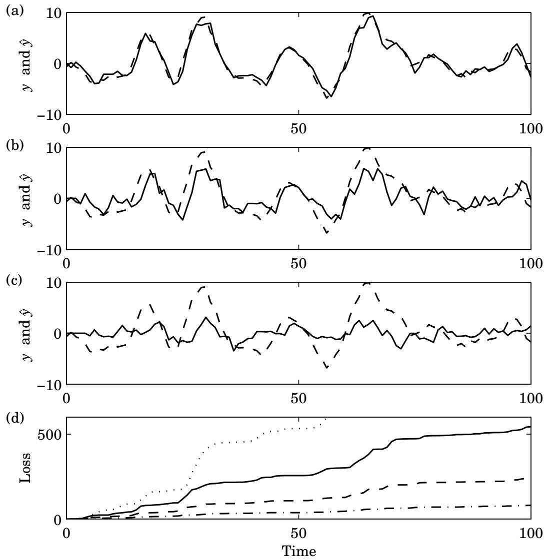
$$\begin{aligned} q^4 &: & 1 &= 1 & & \\ q^3 &: & -0.2 &= -1.5 + f_1 & f_1 &= 1.3 \\ q^2 &: & 0.5 &= 0.7 - 1.5f_1 + f_2 & f_2 &= 1.75 \\ q^1 &: & 0 &= 0.7f_1 - 1.5f_2 + g_0 & g_0 &= 1.715 \\ q^0 &: & 0 &= 0.7f_2 + g_1 & g_1 &= -1.225 \end{aligned}$$

$$\hat{y}(k+3|k) = \frac{qG(q)}{C(q)}y(k) = \frac{1.715q^2 - 1.225q}{q^2 - 0.2q + 0.5}y(k)$$

$$\mathrm{E}\tilde{y}^2 = 1 + 1.3^2 + 1.75^2 = 5.7525$$

Prediction and loss

Output $y(k)$ (dashed), predicted output $\hat{y}(k|k-m)$ (full)
a) $m = 1$, b) $m = 3$, c) $m = 5$
d) Accumulated loss $\sum(y(k) - \hat{y}(k|k-m))^2$ ($m = 1$ dash-dotted, $m = 2$ dashed, $m = 3$ full, $m = 5$ dotted)

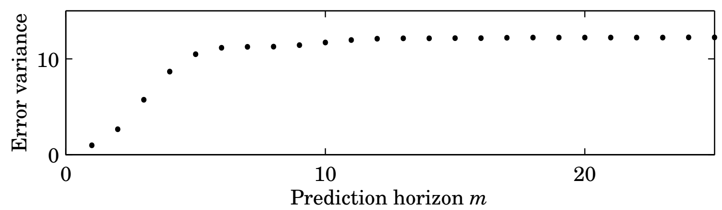


How far is meaningful to predict?

$$y(k) = \frac{C(q)}{A(q)}\, e(k) = \frac{q^2 - 0.2q + 0.5}{q^2 - 1.5q + 0.7}\, e(k)$$
$$= \Big(1 + 1.3q^{-1} + 1.75q^{-2} + 1.715q^{-3} + 1.348q^{-4} + \dots\Big)\, e(k) = \sum_{j=0}^{\infty} f_j q^{-j}\, e(k)$$

The f_i's do not change with m (but G does).

The prediction loss is $\mathrm{E}\tilde{y}^2 = \sigma^2 \sum_{j=0}^{m-1} f_j^2$



Solution of the identity

$$q^{m-1}C(q) = A(q)F(q) + G(q)$$

Compare with the Diophantine equation

$$\begin{aligned}
c_1 &= a_1 + f_1 \\
c_2 &= a_2 + a_1 f_1 + f_2 \\
&\dots \\
c_{m-1} &= a_{m-1} + a_{m-2} f_1 + \dots + a_1 f_{m-2} + f_{m-1} \\
c_m &= a_m + a_{m-1} f_1 + \dots + a_1 f_{m-1} + g_0 \\
c_{m+1} &= a_{m+1} + a_m f_1 + \dots + a_2 f_{m-1} + g_1 \\
&\dots \\
c_n &= a_n + a_{n-1} f_1 + \dots + a_{n-m+1} f_{m-1} + g_{n-m} \\
0 &= a_n f_1 + a_{n-1} f_2 + \dots + a_{n-m+2} f_{m-1} + g_{n-m+1} \\
&\dots \\
0 &= a_n f_{m-1} + g_{n-1} \qquad \text{Can be solved recursively!}
\end{aligned}$$

C with zeros on the unit circle

$$y(k) = e(k) - e(k-1) = \frac{q-1}{q}\, e(k)$$

Formal computation

$$\hat{y}(k+1|k) = -\frac{q}{q-1}\, y(k)$$

Calculate $e(k)$ from $y(k), y(k-1), \dots, y(k_0)$

$$e(k) = e(k-1) + y(k) = \dots = e(k_0 - 1) + \sum_{i=k_0}^{k} y(i)$$

Influence of initial condition!

Kalman filter gives time-varying predictor

$$\hat{y}(k+1|k) = -K(k)(y(k) - \hat{y}(k|k-1)) \quad \text{where} \quad K(k) \quad \to 1$$

Summary – Prediction

- Model C stable

$$y(k) = \frac{C(q)}{A(q)}\, e(k) = \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k)$$

- Identity $\quad q^{m-1}C(q) = A(q)F(q) + G(q)$
- Predictor

$$\hat{y}(k+m|k) = \frac{G^*(q^{-1})}{C^*(q^{-1})}\, y(k) = \frac{qG(q)}{C(q)}\, y(k)$$

- Prediction error

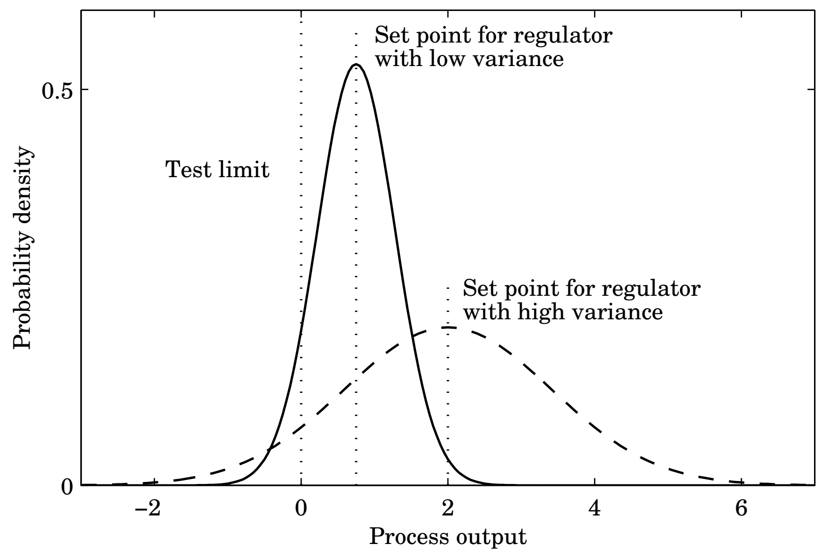
$$\tilde{y}(k+m|k) = F(q)e(k+1)$$

- Optimal predictor dynamics $C(q)$

Minimum variance control

- Motivation
- Problem formulation
- Minimum variance control
- Zeros outside the unit circle
- Summary

Motivation



Problem formulation

- Process model

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

$\deg A - \deg B = d$, $\deg C = n$
C stable
SISO, Innovation model

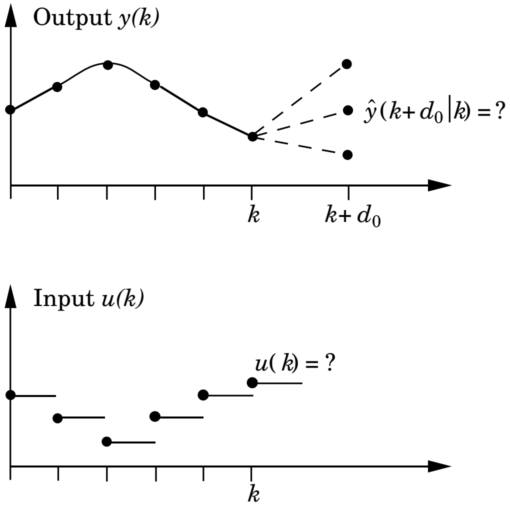
- Design criteria: Minimize

$$\mathrm{E}y^2$$

under the condition that the closed loop system is stable

- May assume any causal nonlinear controller

Minimum variance control = Prediction



Choose $d_0 = d$ and $u(k)$ such that $\hat{y}(k+d|k) = 0$!

Derivation of MV controller

System with stable inverse

$$y(k) = \frac{B(q)}{A(q)}\,u(k) + \frac{C(q)}{A(q)}\,e(k)$$

Rewrite the output d steps ahead

$$\begin{aligned} y(k+d) &= q^d \frac{C(q)}{A(q)}\,e(k) + q^d \frac{B(q)}{A(q)}\,u(k) \\ &= \underbrace{F(q)e(k+1)}_{\tilde{y}(k+d|k)} + \underbrace{\frac{qG(q)}{A(q)}\,e(k) + \frac{q^d B(q)}{A(q)}\,u(k)}_{\hat{y}(k+d|k)} \end{aligned}$$

Derivation cont'd

Substitute the expression for old innovations

$$e(k) = \frac{A(q)}{C(q)}\,y(k) - \frac{B(q)}{C(q)}\,u(k)$$

$$\begin{aligned} y(k+d) &= F(q)e(k+1) + \frac{q^d B(q)}{A(q)}\,u(k) \\ &\quad + \frac{qG(q)}{A(q)} \left\{ \frac{A(q)}{C(q)}\,y(k) - \frac{B(q)}{C(q)}\,u(k) \right\} \\ &= F(q)e(k+1) + \frac{qG(q)}{C(q)}\,y(k) \\ &\quad + \frac{qB(q)}{C(q)} \left\{ \frac{q^{d-1}C(q)}{A(q)} - \frac{G(q)}{A(q)} \right\} u(k) \\ &= \underbrace{F(q)e(k+1)}_{\tilde{y}(k+d|k)} + \underbrace{\frac{qG(q)}{C(q)}\,y(k) + \frac{qB(q)F(q)}{C(q)}\,u(k)}_{\hat{y}(k+d|k)} \end{aligned}$$

Derivation cont'd

$$y(k+d) = \underbrace{F(q)e(k+1)}_{\tilde{y}(k+d|k)} + \underbrace{\frac{qG(q)}{C(q)}\,y(k) + \frac{qB(q)F(q)}{C(q)}\,u(k)}_{\hat{y}(k+d|k)}$$

$u(k)$ is function of $y(k), y(k-1), \ldots$ and $u(k-1), u(k-2), \ldots$, so

$$\mathrm{E}y^2(k+d) = \mathrm{E}\big(\tilde{y}(k+d|k)\big)^2 + \mathrm{E}\big(\hat{y}(k+d|k)\big)^2$$

and

$$\mathrm{E}y^2(k+d) \geq \big(1 + f_1^2 + \cdots + f_{d-1}^2\big)\sigma^2$$

Equality is obtained for $\hat{y}(k+d|k) = 0$,

$$u(k) = -\frac{G(q)}{B(q)F(q)}\,y(k)$$

Minimum variance controller

Some remarks

- Still true if a linear controller is postulated and if $e(k)$ and $e(j)$ are uncorrelated
- Resulting controller implies

$$\hat{y}(k+d|k) = 0$$

- The control error is a moving average of order $d-1$

$$y(k) = \tilde{y}(k|k-d) = \frac{F(q)}{q^{d-1}}\, e(k)$$

- All process zeros, $B(q)$, are canceled

Example MV-control

$A(q) = q^3 - 1.7q^2 + 0.7q \qquad B(q) = q + 0.5 \qquad C(q) = q^3 - 0.9q^2$

The identity

$$q(q^3 - 0.9q^2) = (q^3 - 1.7q^2 + 0.7q)(q + f_1) + g_0q^2 + g_1q + g_2$$

Equate coefficients

$$\begin{aligned}
q^3 &: \quad -0.9 = f_1 - 1.7 & f_1 &= 0.8 \\
q^2 &: \quad 0 = -1.7f_1 + 0.7 + g_0 & g_0 &= 0.66 \\
q^1 &: \quad 0 = 0.7f_1 + g_1 & g_1 &= -0.56 \\
q^0 &: \quad 0 = g_2 & g_2 &= 0
\end{aligned}$$

Controller $u(k) = -\dfrac{G(q)}{F(q)B(q)}y(k) = -\dfrac{0.66q^2 - 0.56q}{(q+0.8)(q+0.5)}y(k)$

Output $y(k) = e(k) + 0.8e(k-1) \qquad r_y(\tau) = ?$

Example – Influence of delay

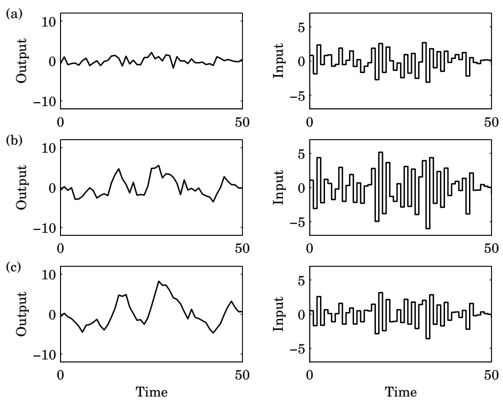
What is the performance of the MV controller when

$A(q) = q^{d-1}(q^2 - 1.5q + 0.7)$

$B(q) = q + 0.5$

$C(q) = q^{d-1}(q^2 - 0.2q + 0.5)$

when $d = 1$, 3, and 5?



Closed loop poles

MV controller defined by

$$u(k) = -\frac{S(q)}{R(q)}y(k) = -\frac{G(q)}{F(q)B(q)}y(k)$$

Closed loop characteristic polynomial

$$\begin{aligned}
A(q)R(q) + B(q)S(q) &= A(q)B(q)F(q) + B(q)G(q) \\
&= B(q)q^{d-1}C(q)
\end{aligned}$$

What if B unstable? Look at the control signal

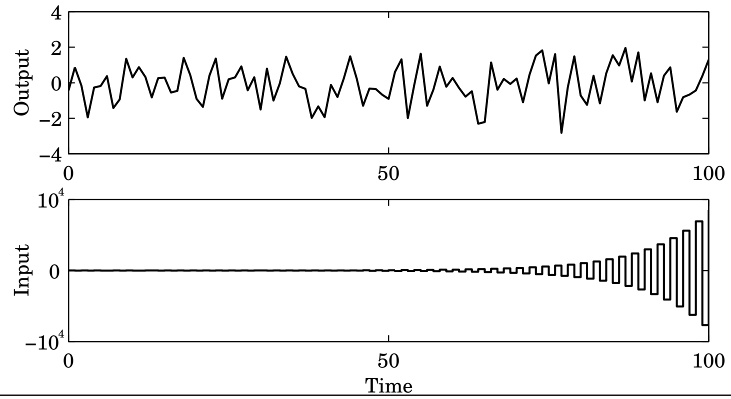
$$u(k) = -\frac{G(q)}{B(q)F(q)}y(k) = -\frac{G(q)}{q^{d-1}B(q)}e(k)$$

Example – Unstable inverse

$A(z) = (z-1)(z-0.7) \qquad B(z) = 0.9z+1 \qquad C(z) = z(z-0.7)$

Zero at $z = -10/9$. Direct calculation of MV controller gives

$$u(k) = -\frac{q-0.7}{0.9q+1}y(k) \qquad \mathrm{E}y^2 = \sigma^2$$



MV control, general case

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$
$$B(q) = B^+(q)B^-(q)$$

The minimum variance controller is given by

$$u(k) = -\frac{G(q)}{B^+(q)F(q)}y(k)$$

$$q^{d-1}C(q)B^{-*}(q) = A(q)F(q) + B^-(q)G(q)$$
$$\deg F = d + \deg B^- - 1$$
$$\deg G < \deg A = n$$

with monic reciprocal polynomial

$$B^{-*}(q) = q^{\deg B^-}B^-(q^{-1})$$

MV control, general case cont'd

Control error

$$y(k) = \frac{F(q)}{q^{d-1}B^{-*}(q)}e(k)$$

Closed loop characteristic polynomial

$$A(q)R(q) + B(q)S(q) = B^+(q)q^{d-1}C(q)B^{-*}(q)$$

Reflect the unstable zeros in the unit circle!

Example – Unstable inverse cont'd

The Diophantine equation becomes

$$z(z-0.7)(z+0.9) = (z-1)(z-0.7)(z+f_1) + (0.9z+1)(g_0z+g_1)$$

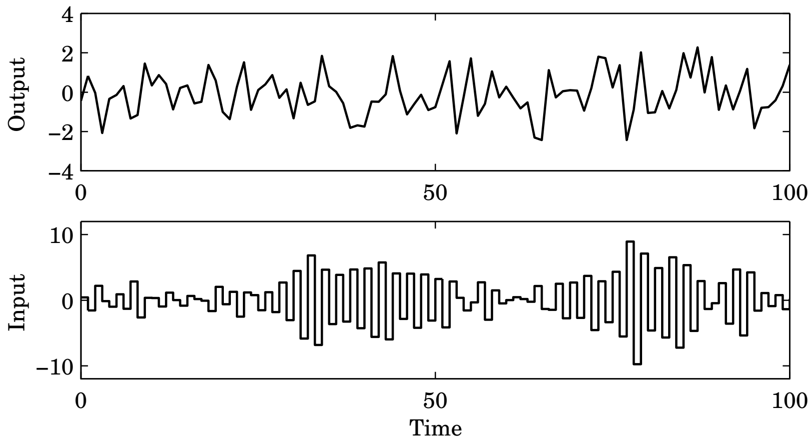
Solution

$$f_1 = 1 \qquad g_0 = 1 \qquad g_1 = -0.7$$
$$y(k) = \frac{q+1}{q+0.9}e(k) = e(k) + \frac{0.1}{q+0.9}e(k)$$

Output variance

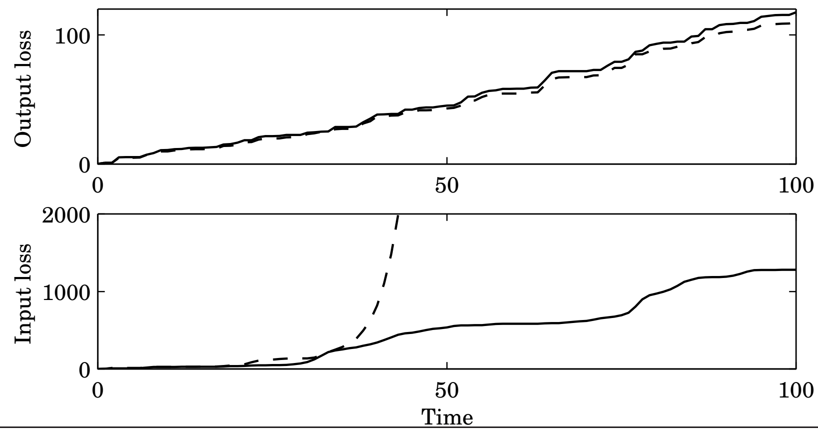
$$\mathrm{E}y^2 = \sigma^2 + \frac{0.1^2}{1-0.9^2}\sigma^2 = \frac{20}{19}\sigma^2 \qquad (+5\%)$$

Example – Unstable inverse cont'd



Comparison

Accumulated loss of output and input
Unstable MV (dashed)
Stable MV (full)



Summary

- Minimum variance control is of practical relevance
- MV control by Prediction
- Interpretation as pole-placement

$$q^{d-1}C(q)B^+(q)B^{-*}(q) = A(q)B^+(q)F(q) + B(q)G(q)$$

- Reflect the unstable zeros in the unit circle!
- LQG?
- Reference signals?