STOCHASTIC CONTROL

- WHY?
- HOW?
- WHAT TO DO?

MY LEARNING PHILOSOPHY

- GET THE MAIN IDEAS
- DISCUSS
- WORK YOURSELF

DAGENS IDÉ

Utan stokastik –

No worries

- ÅSTRÖM (1970): INTRODUCTION TO STOCHASTIC CONTROL
- SOME COPIES OF ARTICLES AND CHAPTERS
- GOOD EXAMPLES
  SOLUTIONS DO EXIST, BUT WHERE
- MATLAB HELPS TO GET THE INTUITION
- HOME PAGE
- WHAT DO YOU KNOW?

Tsypkin ?

New ideas:

1. Counterintuitive
2. Make sense
3. Trivial

Optimal reglering av <u>deterministiska</u> system:

* Ingen skillnad mellan öppen och sluten styrning
* Optimala regulatorn $\mathcal{S} \to \mathcal{U}$

Ingen dynamik

Exempel

$$\frac{dx}{dt} = u$$

$$x(0) = 1$$

$$J = \int_0^\infty (x^2 + u^2)\,dt$$

$J_{min} = 1$ fås för

$u(t) = -e^{-t}$ openloop (1)

$u(t) = -x(t)$ closed loop (2)

(1) $\Rightarrow$ stabilt system

(2) $\Rightarrow$ asymp. stab. system

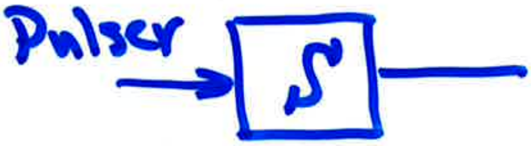
Störningskänsligheten!

HUR BESKRIVA STÖRNINGAR?

* Deterministiska störningar

  Steg
  Puls
  Sinus

* Styckvis determ. störn

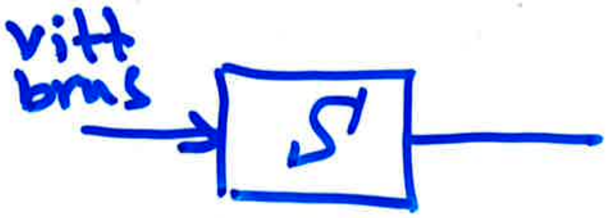


* Singulära stokastiska proc

$$x(t) = \sum a_i(t)\,\xi_i$$

stok var
känd tidsfunk

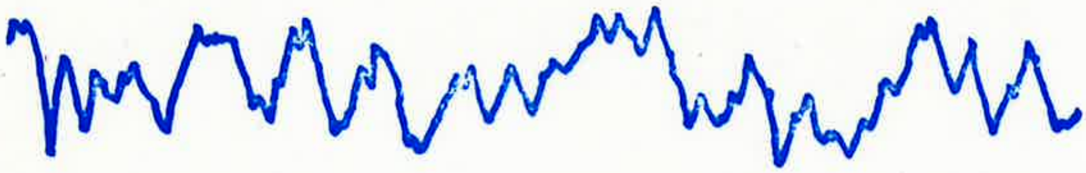
* Stokastiska processer



HISTORIK

R. BROWN 1827 Observation

A. EINSTEIN 1905 Analys



LANGEVIN, FOKKER, PLANCK, JOHNSSON

DOOB, WIENER, CRAMER, LEVY

KHINCHINE, KOLMOGOROV,

LANIN BATTIN

KALMAN BUCY

RADARBRUS

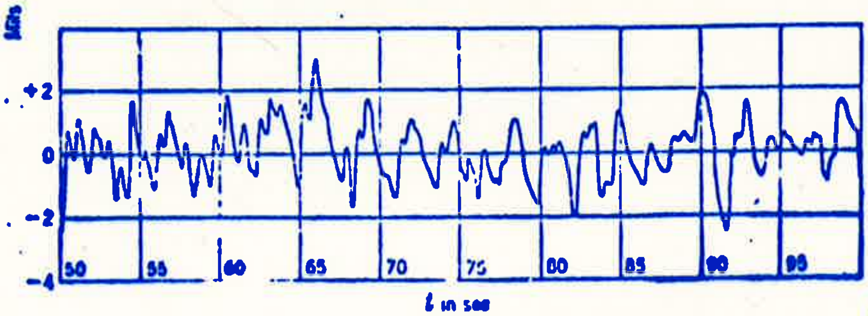
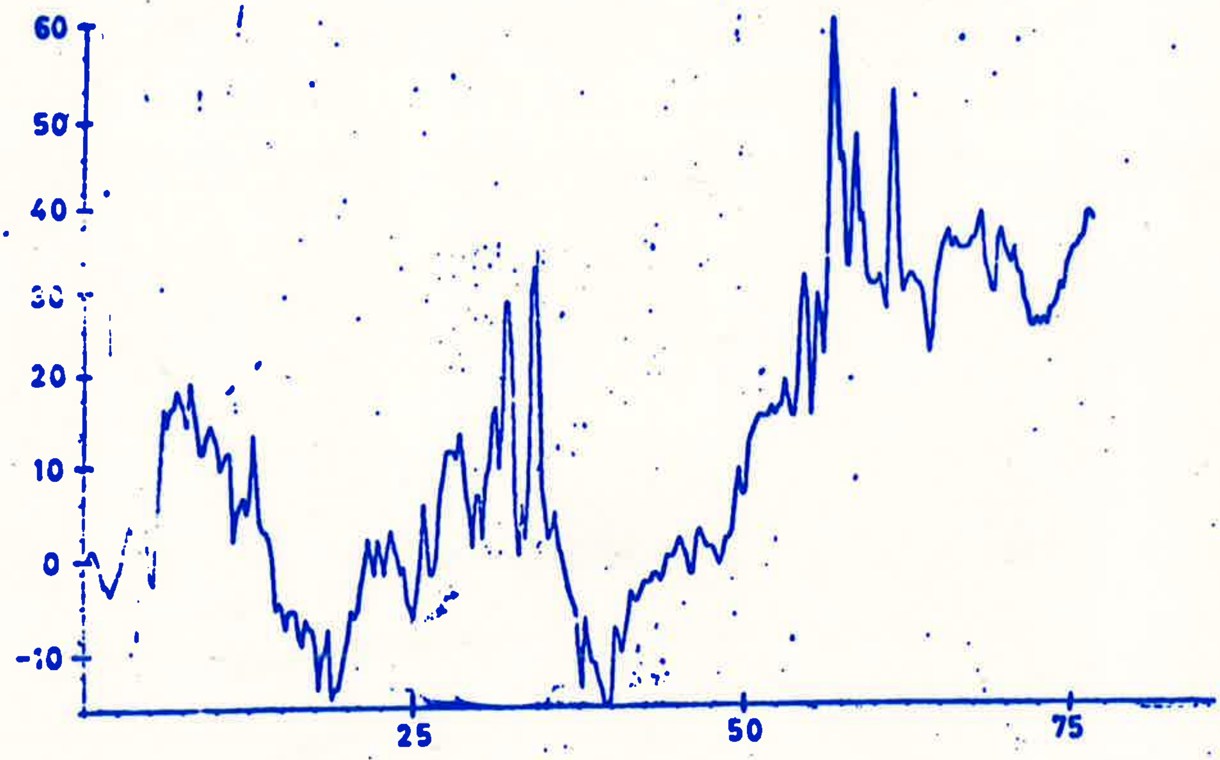


Fig. 6-8.—Traverse tracking data, outgoing radial course.

GYRODRIFT



KÖLÄNGD

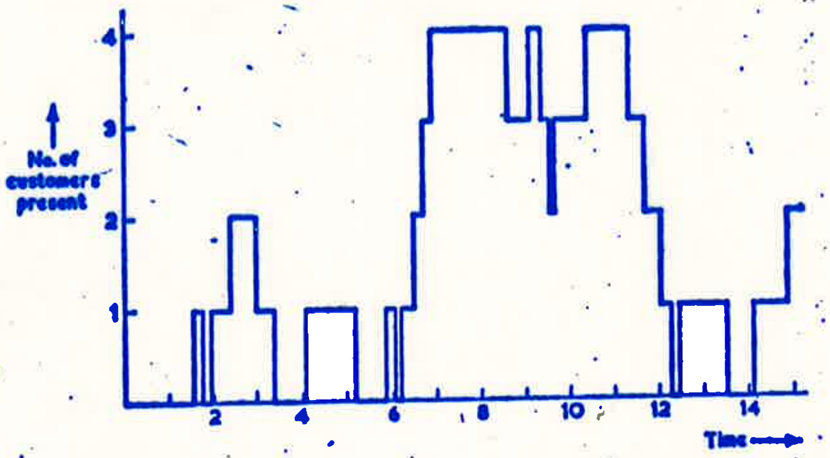


Fig. 1.2. Number of customers in system for realization of Table 1.2.

YTVIKTSVARIATIONER

STOKASTISKA PROCESSER

STOKASTISKA VARIABLER

DEF. AV STOK. PROCESSER

KARAKTÄRISERING

SPECIELLA PROCESSER

SPEKTRALTÄTHET

VITT BRUS

ELEMENTA

En stokastisk variabel är en funktion definierad på ett sannolikhetsfält $X: \Omega \to R$

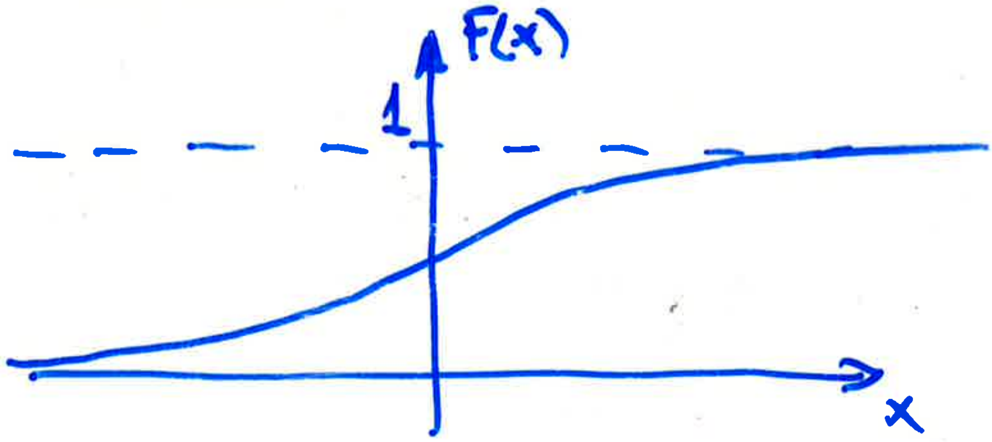
ω elementarhändelse

Ω utfallsrum

* Fördelningsfunktion

$$F(\xi) = P(X(\omega) \le \xi)$$

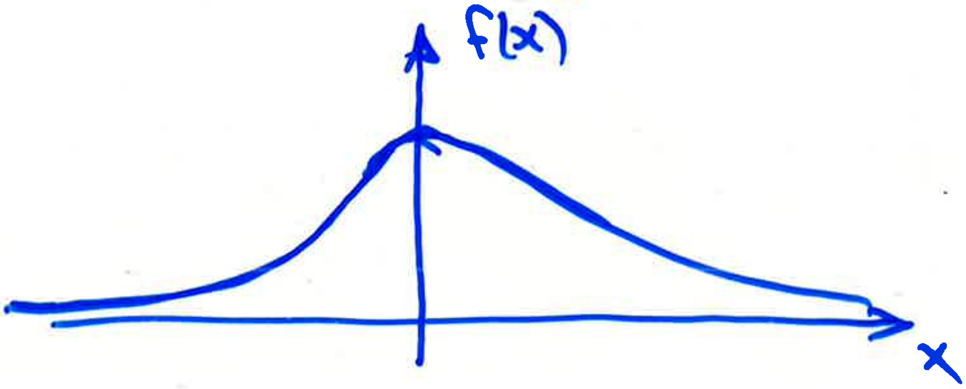
- F ökande
- $\lim_{x \to \infty} F(x) = 1$
- $\lim_{x \to -\infty} F(x) = 0$



* Frekvensfunktion (Täthetsfunktion)

$$f(\xi) = \frac{dF(\xi)}{d\xi}$$

- $p(x) \geq 0$
- $\int_{-\infty}^{\infty} p(x)\,dx = 1$
- $P(a \leq \xi \leq b) = \int_a^b p(x)\,dx$



* Normalfördelning $N(m, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}}$$

* Medelvärde

$m = \bar{x} = Ex$

* Varians

$E(x-m)^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$

FLERVARIABEL STOK. VAR.

$$F(\xi_1, \xi_2) = P(x_1(w) \leq \xi_1, x_2(w) \leq \xi_2)$$

Frekvensfunktion

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}$$

"Sandhög"

x_1 & x_2 oberoende om

$$f(x_1, x_2) = f_1(x_1) f_2(x_2)$$

Kovarians

$$r_{12} = \operatorname{cov}(x_1, x_2) = E[(x_1 - m_1)(x_2 - m_2)]$$

Korrelation

$$\rho_{12} = \frac{r_{12}}{\sqrt{r_1 r_2}}$$

$$f(x) = \frac{1}{(2\pi)^{n/2} (\det R)^{1/2}} e^{-\frac{(x-m)^T R^{-1} (x-m)}{2}}$$

STOKASTISK PROCESS

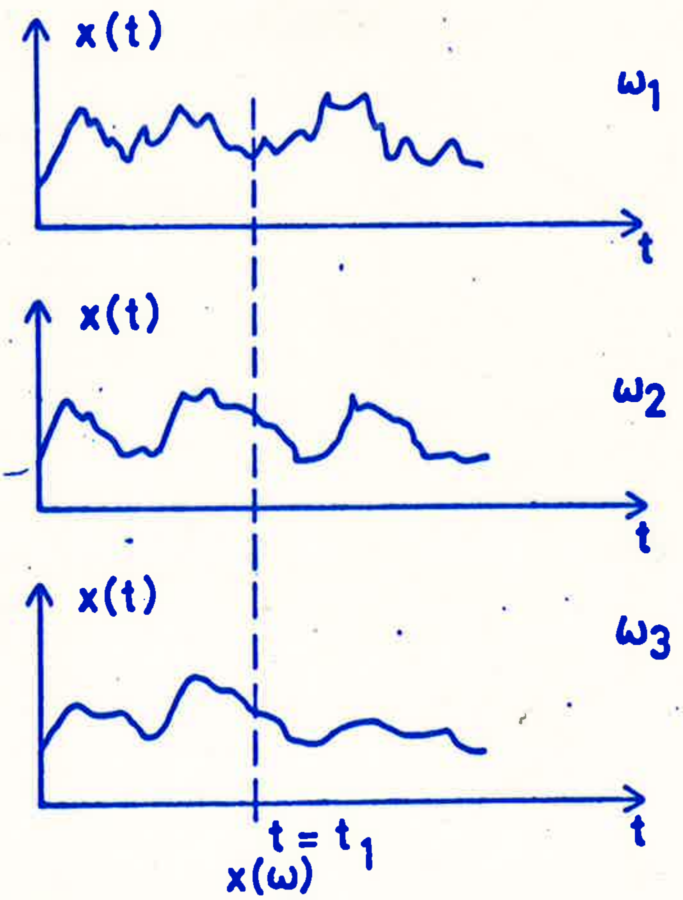
$$\{x(t,\omega)\} = \{x(t,\omega),\ t \in T,\ \omega \in \Omega\}$$

Ω : UTFALLSRUM

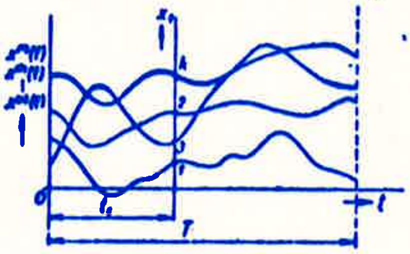
ω : ELEMENTARHÄNDELSE

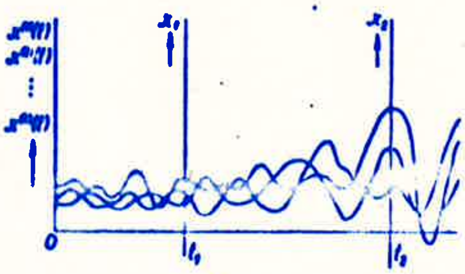
ω fix $\Longrightarrow$ $x(t, \omega_1)$ EN VANLIG FUNKTION UTFALL ELLER REALISATION

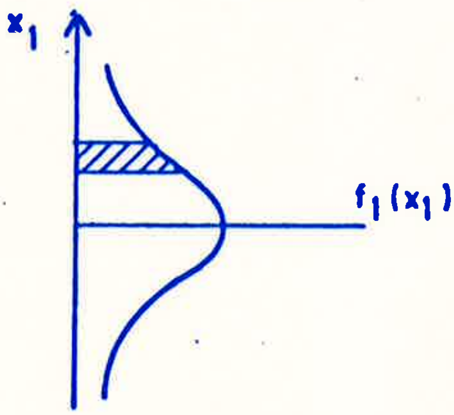
t fix $\Longrightarrow$ $x(t_1, \omega)$ STOKASTISK VARIABEL



HUR BESKRIVS EN STOKASTISK PROCESS







FREKVENSFUNKTION

$$f_t(x)$$

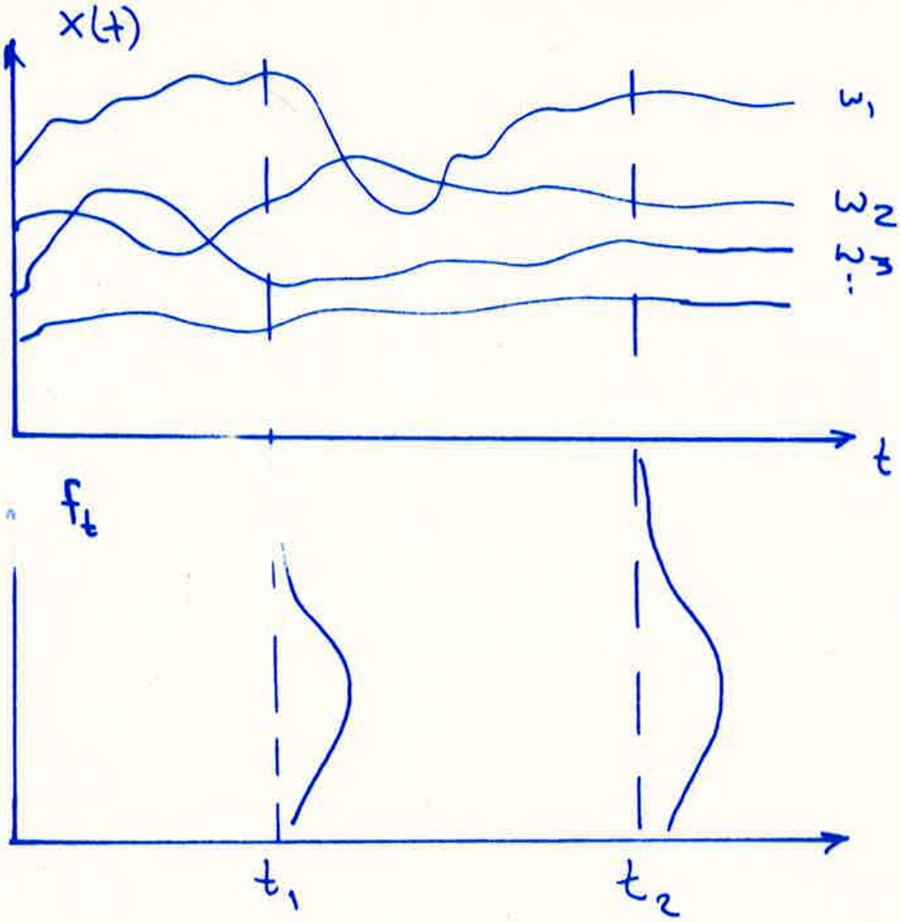
MEDELVÄRDE

$$m(t_1) = Ex(t_1) = \int_{-\infty}^{\infty} x \, f_{t_1}(x) \, dx$$

KOVARIANS

$$r(t_1, t_2) = E\left[x(t_1) - m(t_1)\right]\left[x(t_2) - m(t_2)\right]$$

$$= \int_{-\infty}^{\infty} (x_1 - m_1)(x_2 - m_2) \, f_{t_1 t_2}(x_1 \, x_2) \, dx_1 \, dx_2$$



STATIONÄR PROCESS

SVAGT STATIONÄR PROCESS

$$m(t) = m$$

$$r(t_2, t_1) = r(t_2 - t_1)$$

ERGODISK PROCESS

$$E\,x(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t, \omega)\, dt$$

(nästan alla w)

Ex

Ej ergodisk process

$$x(t) = x(0)$$

$$x(0) = \begin{cases} 1 & \text{slh } 0.5 \\ -1 & \text{slh } 0.5 \end{cases}$$

$$E[x] = 0$$

Tidsmedelv +1 eller -1

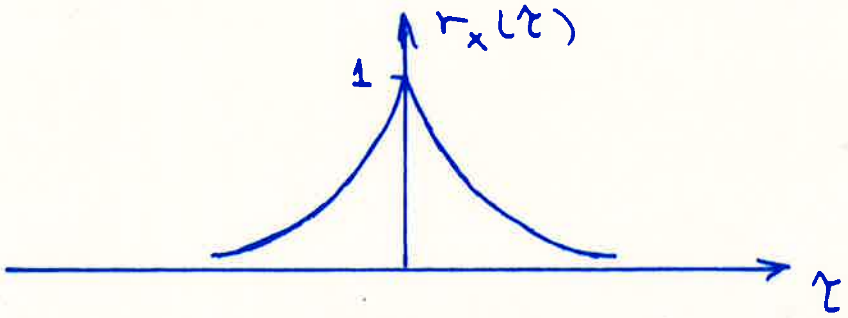
KOVARIANSFUNKTION

$$r_x(s,t) = r_x(s-t) = E\big(x(s)-m_x(s)\big)\big(x(t)-m_x(t)\big)$$

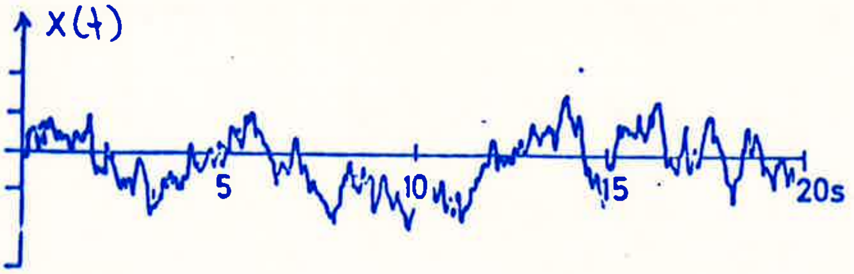
1) $r_x(\tau) = r_x(-\tau)$

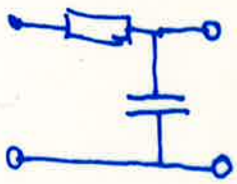
2) $r_x(0) \geq |r_x(\tau)|$

Givet $r_x(\tau) = e^{-a|\tau|}$

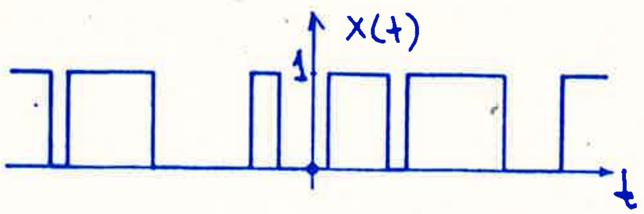


Hur ser $x(t)$ ut?





NORMAL



RANDOM TELEGRAPH WAVE

$r(\tau)$ - EGENSKAPER

* $r(\tau) = r(-\tau)$
* $\sum z_i z_j r(t_i - t_j) \geq 0$
* $|r(\tau)| \leq r(0)$
* $r(\tau)$ kont för $\tau = 0 \Rightarrow$
  $r(\tau)$ kont för $\forall \tau$
* $r(\tau) = r(0)$ för ngt τ
  $\Rightarrow r(\tau)$ periodisk

SPEKTRALTÄTHET

KONT TID

$$\begin{cases} \phi(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-i\omega\tau} r(\tau)\, d\tau \\ r(\tau) = \displaystyle\int_{-\infty}^{\infty} e^{i\omega\tau} \phi(\omega)\, d\omega \end{cases}$$

$$\mathrm{Var}\, x = r_x(0) = \int_{-\infty}^{\infty} \phi(\omega)\, d\omega$$

DISKRET TID

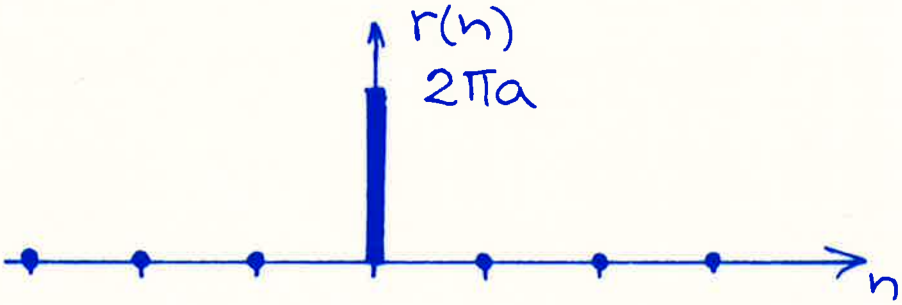
$$\begin{cases} \phi(\omega) = \dfrac{1}{2\pi} \displaystyle\sum_{n=-\infty}^{\infty} r(n)\, e^{-in\omega} \\ r(n) = \displaystyle\int_{-\pi}^{\pi} e^{in\omega} \phi(\omega)\, d\omega \end{cases}$$

VITT BRUS

DISKRET TID

$$\phi(\omega) = a$$

$$r(n) = \int_{-\pi}^{\pi} e^{in\omega} \cdot a\, d\omega = \frac{2a}{n} \sin n\pi$$



KONTINUERLIG TID

$$\phi(\omega) = a$$

$$r(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} a\, d\omega$$

BANDBEGRÄNSAT VITT BRUS

$$\phi(\omega) = \begin{cases} a & |\omega| < \Omega \\ 0 & |\omega| \geq \Omega \end{cases}$$

NÄR $\Omega \to \infty$

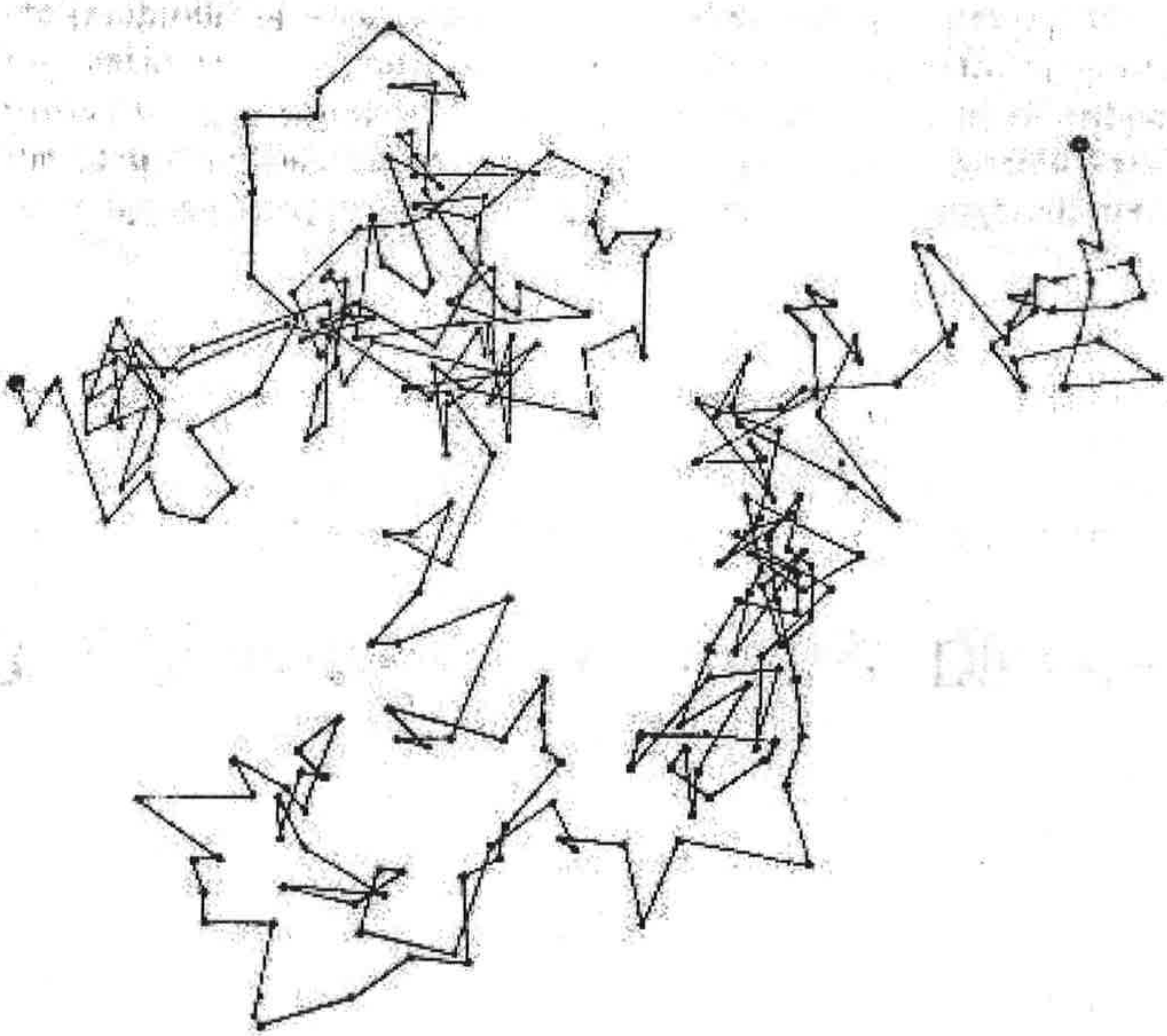
$$r(\tau) = 2\pi a\, \delta(\tau)$$

VARFÖR VITT BRUS?

SPECIELLA STOKASTISKA PROCESSER

- NORMAL PROCESS
- PROCESS AV ANDRA ORDNINGEN
- VITT BRUS
- WIENER PROCESS

Brownian motion



WIENER PROCESS

BROWNSK RÖRELSE (1827)

EINSTEIN (1905)

$x(t)$ - koordinat för partikel

Def

1. $x(0)=0$
2. $x(t)$ normal
3. $Ex(t)=0 \quad \forall t>0$
4. Oberoende stationära inkrement

Egenskaper

$$m(t) = Ex(t) = 0$$

$$\text{var } x(t) = c \cdot t$$

$$r(s,t) = \text{cov}[x(s), x(t)] = c \min(s,t)$$

Samplefunktionen av WP

- kontinuerlig m slh 1
- derivatan $\nexists$ någonstans
- har oändlig längd

ANALYS AV STOK. PROC.

Hur definieras

- konvergens
- kontinnuitet
- derivata
- integral

?

KONVERGENS

$$\{x_n(\omega),\ n=1,2,\ldots\}$$

- Slh 1

  $$P\{\omega;\ x_n(\omega) \to x(\omega)\} = 1$$

  $\Downarrow$

  utom för ω med slh måttet 0

- I sannolikhet

  $\Uparrow$

  $$\lim_{n\to\infty} P\{\omega;\ |x_n(\omega) - x(\omega)| \geq \varepsilon\} = 0 \quad \forall\varepsilon$$

- I medelkvadratmening

  $$\lim_{n\to\infty} E|x_n - x|^2 = 0$$

Th

$\{x_n\}$ svit av stok. var

$E x_n^2 < \infty$

$x_n \to x$ i medelkvadrat

$$\Rightarrow \lim_{n \to \infty} E x_n = E \lim_{n \to \infty} x_n = E x$$

Kontinuitet

$\{x(t)\}$ kont i medelkv. mening om

$$\lim_{h \to 0} E[x(t+h) - x(t)]^2 = 0$$

Th

$\{x(t)\}$ kont $\iff$ 1) $m(t)$ kont
(2:a ordn) 2) $r(t,t)$ kont

Deriverbarhet

$$\lim_{h \to 0} E\left\{ \frac{x(t_0+h) - x(t_0)}{h} - x'(t_0) \right\}^2 = 0$$

Ex Wiener process

$\{x(t)\}$

$$E[x(t+h)-x(t)]^2 = h$$

$\Rightarrow \{x(t)\}$ kontinuerlig

$$E\left[\frac{x(t+h)-x(t)}{h}\right]^2 = \frac{1}{h}$$

$\Rightarrow \{x(t)\}$ ej der.bar

Kan visa formellt att
Derivatan av en Wiener
process är vitt brus

SAMMANFATTNING

- Måste ha en känsla för vad en stokastisk process är
- Medelvärde, kovariansfkn
- Normal fördelning
- Tidsdiskret lätt
- Tidskontinuerligt svårt
- Kommer att använda normalfördelat brus och medelkvadrat mening