8.3.1. $\quad y = Au + x$

$$\ell = q_0 + q_1^T y + q_2^T u + \frac{1}{2} y^T Q_1 y + y^T Q_{12} u + \frac{1}{2} u^T Q_2 u$$

<u>i) no state information</u>

$$\ell = q_0 + q_1^T A u + q_1^T x + q_2^T u + \tfrac{1}{2}(u^T A^T + x^T) Q_1 (Au + x) + u^T A^T Q_{12} u + x^T Q_{12} u + \tfrac{1}{2} u^T Q_2 u$$

$$E(\ell) = q_0 + q_1^T A u + q_1^T m + q_2^T u + \tfrac{1}{2} u^T A^T Q_1 A u + \tfrac{1}{2} u^T A^T Q_1 m + \tfrac{1}{2} m^T Q_1 A u +$$

$$+ \frac{1}{2} m^T Q_1 m + \tfrac{1}{2} \operatorname{tr} Q_1 R + u^T A^T Q_{12} u + m^T Q_{12} u + \frac{1}{2} u^T Q_2 u =$$

$$= q_0 + q_1^T m + \tfrac{1}{2} m^T Q_1 m + \tfrac{1}{2} \operatorname{tr} Q_1 R + u^T \underbrace{\left(\tfrac{1}{2} A^T Q_1 A + A^T Q_{12} + \tfrac{1}{2} Q_2\right)}_{Q} u +$$

$$+ (q_1^T A + q_2^T + \tfrac{1}{2} m^T Q_1 A + m^T Q_{12}) u + \tfrac{1}{2} u^T A^T Q_1 m =$$

$$= q_0 + q_1^T m + \tfrac{1}{2} m^T Q_1 m + \tfrac{1}{2} \operatorname{tr} Q_1 R + u^T Q u +$$

$$+ \tfrac{1}{2}(q_1^T A + q_2^T + m^T Q_1 A + m^T Q_{12}) u + u^T (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m) \cdot \tfrac{1}{2} =$$

$$= q_0 + q_1^T m + \tfrac{1}{2} m^T Q_1 m + \tfrac{1}{2} \operatorname{tr} Q_1 R +$$

$$+ \left[u + \tfrac{1}{2} Q^{-1} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)\right]^T Q \left[u + Q^{-1} \tfrac{1}{2} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)\right] -$$

$$- \tfrac{1}{4} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)^T Q^{-1} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)$$

$$\underline{E(\ell) \geq q_0 + q_1^T m + \tfrac{1}{2} m^T Q_1 m + \tfrac{1}{2} \operatorname{tr} Q_1 R - \tfrac{1}{4} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)^T Q^{-1} (\qquad)}$$

$$\underline{u = -\tfrac{1}{2} Q^{-1} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)}$$

ii) complete state information

$$\ell = q_0 + q_1^T A u + q_1^T x + q_2^T u + \tfrac{1}{2}(u^T A^T + x^T) Q_1 (Au + x) + (u^T A^T + x^T) Q_{12} u + \tfrac{1}{2} u^T Q_2 u =$$

$$= q_0 + q_1^T x + \tfrac{1}{2} x^T Q_1 x + u^T \underbrace{\left(\tfrac{1}{2} A^T Q_1 A + A^T Q_{12} + \tfrac{1}{2} Q_2\right)}_{Q} u + (q_1^T A + q_2^T + \tfrac{1}{2} x^T Q_1 A + x^T Q_{12}) u +$$

$$+ u^T (\tfrac{1}{2} A^T Q_1 x) =$$

$$= q_0 + q_1^T x + \tfrac{1}{2} x^T Q_1 x + \left[u + Q^{-1}\tfrac{1}{2}(A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)\right]^T Q \left[u + Q^{-1}\tfrac{1}{2}(A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)\right] -$$

$$- \tfrac{1}{4}(A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)^T Q^{-1} (A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)$$

$$\ell \geq q_0 + q_1^T x + \tfrac{1}{2} x^T Q_1 x - \tfrac{1}{4}(A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)^T Q^{-1} (A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)$$

$$\underline{u = -\tfrac{1}{2} Q^{-1} (A^T q_1 + q_2 + A^T Q_1 x + Q_{12}^T x)}$$

$$\underline{E\,\ell = q_0 + q_1^T m + \tfrac{1}{2} m^T Q_1 m + \tfrac{1}{2} \mathrm{tr}\, Q_1 R - \tfrac{1}{4}(A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m)^T Q^{-1} (A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m) -}$$

$$\underline{- \tfrac{1}{4} \mathrm{tr}\, (A^T Q_1 + Q_{12}^T) Q^{-1} (A^T Q_1 + Q_{12}^T) R}$$

8.3.2. $A = \begin{pmatrix} 0.666 & -0.188 & 0.671 \\ -0.052 & -0.296 & 0.259 \\ 0.285 & 2.358 & -1.427 \end{pmatrix} \qquad m = \begin{pmatrix} -5.39 \\ -3.704 \\ -0.729 \end{pmatrix}$

$$q_0 = 0 \quad q_1 = 0 \quad q_2 = 0 \quad Q_1 = I \quad Q_2 = I \quad Q_{12} = 0$$

$$u = -\tfrac{1}{2}\left(\tfrac{1}{2}A^T Q_1 A + A^T Q_{12} \;\; \tfrac{1}{2}Q_2\right)^{-1}\left(A^T q_1 + q_2 + A^T Q_1 m + Q_{12}^T m\right) =$$

$$= -\tfrac{1}{2}\left(\tfrac{1}{2}A^T A + \tfrac{1}{2} I\right)^{-1}\left(A^T m\right) =$$

$$= -\left(\begin{bmatrix} 0.666 & -0.052 & 0.285 \\ -0.188 & -0.296 & 2.358 \\ 0.671 & 0.259 & -1.427 \end{bmatrix}\begin{bmatrix} 0.666 & -0.188 & 0.671 \\ -0.052 & -0.296 & 0.259 \\ 0.285 & 2.358 & -1.427 \end{bmatrix} + I\right)^{-1}\begin{bmatrix} 0.666 & -0.052 & 0.285 \\ 0.188 & -0.296 & 2.358 \\ 0.671 & 0.259 & -1.427 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} -5.39 \\ -3.704 \\ -0.729 \end{bmatrix} = -\begin{bmatrix} 1.527485 & 0.562214 & 0.026723 \\ 0.562214 & 6.683124 & -3.567678 \\ 0.026723 & -3.567678 & 3.553651 \end{bmatrix}^{-1}\begin{bmatrix} 3.604897 \\ -1.635918 \\ -3.535743 \end{bmatrix} =$$

$$= \frac{-1}{15.59943}\begin{bmatrix} 11.02116 & -2.09325 & -2.18439 \\ -2.09325 & 5.42743 & 5.46460 \\ -2.18439 & 5.46460 & 9.89229 \end{bmatrix}\begin{bmatrix} 3.604897 \\ -1.635918 \\ -3.535743 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -3.262 \\ +2.292 \\ +3.320 \end{bmatrix}}}$$

8.4.1.

$$x(t+1) = x(t) + u(t) + v(t)$$

$$\ell = \sum_{k=1}^{N} x^2(k) + u^2(k)$$

$$u(t) = -L(t)x(t)$$

$$L(t) = [1 + S(t+1)]^{-1} S(t+1)$$

$$S(t) = S(t+1) + 1 - S(t+1)[1+S(t+1)]^{-1} S(t+1) = \frac{(1+S(t+1))^2 - S^2(t+1)}{1+S(t+1)} = \frac{1+2S(t+1)}{1+S(t+1)}$$

$$S(N) = 1$$

$$\min E\ell = m^T S(1) m + \operatorname{tr} S(1) R_0 + \sum_{s=1}^{N-1} \operatorname{tr} R_1 S(s+1) = m^2 S(1) + \sigma^2 S(1) + \sum_{s=1}^{N-1} r S(s+1)$$

$\underline{t \to \infty}$

$$S(t) = \frac{1+2S(t)}{1+S(t)}$$

$$S^2 - S - 1 = 0 \quad \Rightarrow S(t) = \frac{1}{2} \pm \sqrt{\frac{1}{4}+1} = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$L(t) = \frac{\frac{1}{2}(1+\sqrt{5})}{1+\frac{1}{2}(1+\sqrt{5})} = \underline{\frac{1+\sqrt{5}}{3+\sqrt{5}}}$$

$$\lim_{N\to\infty} E\ell = \infty$$

(Bruset ger ett ständigt bidrag till förlustfunktionen. Om däremot $v(t)=0$ antar $\lim_{N\to\infty} E\ell$ ett ändligt värde.)

8.4.2. $x(t+1) = ax(t) + bu(t) + v(t)$

$$\ell = \sum_{t=1}^{N} x^2(t) = x^2(N) + \sum_{t=1}^{N-1} x^2(t)$$

$u(t) = -L(t)x(t)$

$L(t) = [b\,S(t+1)b]^{-1} b\,S(t+1)a = \underline{\underline{\frac{a}{b}}}$

$S(t) = a^2S(t+1) + 1 - ab\,S(t+1)[b^2S(t+1)]^{-1} b\,S(t+1)\,a = a^2S(t+1) + 1 - a^2S(t+1) = \underline{1}$

$\underline{\min E\ell = m^2 + \sigma^2 + (N-1)r}$

8.4.3. $x(t+1) = ax(t) + bu(t) + v(t)$

$$\ell = \sum_{t=1}^{N} x^2(t) = x^2(1) + \sum_{t=1}^{N-1} x^2(t+1) = x^2(1) + \sum_{t=1}^{N-1}[ax(t) + bu(t) + v(t)]^2 =$$

$$= x^2(1) + \sum_{t=1}^{N-1}(ax(t) + bu(t))^2 + 2\sum_{t=1}^{N-1}(ax(t) + bu(t))v(t) + \sum_{t=1}^{N-1} v(t)^2 =$$

$$= x^2(1) + (ax(1) + bu(1))^2 + \sum_{t=2}^{N-1}(ax(t) + bu(t))^2 + 2\sum(\ ) + \sum v^2 =$$

$$= x^2(1) + (ax(1) + bu(1))^2 + \sum_{t=1}^{N-2}[a^2x(t) + abu(t) + av(t) + bu(t+1)]^2 + 2\sum(\ ) + \sum v^2$$

Välj $\underline{u(t+1) = -\frac{1}{b}(a^2x(t) + abu(t))}$

$\underline{u(1)} = -\frac{a}{b}x(1) = \underline{-\frac{a}{b}m}$

$$\min E\ell = m^2 + R_0 + a^2R_0 \overset{?}{+} \sum_{1}^{N-2} a^2R_1 + \sum_{1}^{N-1} R_1 = \underline{m^2 + (1+a^2)R_0 + (1+a^2)R_1(N-2) + R_1}$$

8.5.1. $\begin{cases} x(t+1) = x(t) + u(t) + v(t) \\ y(t) = x(t) + e(t) \end{cases}$ $\quad v \in N(0, \sqrt{r_1})$

$e \in N(0, \sqrt{r_2})$

$x(1) \in N(m, \sigma)$

$$l = \sum_{k=1}^{N} x^2(k) + q u^2(k)$$

a) $r_1 = r_2 = 0 \Rightarrow$ *Fullständig tillståndskännedom*

$$u(t) = -L(t)x(t) = -L(t)\,y(t)$$

$$L(t) = [q + S(t+1)]^{-1} S(t+1)$$

$$S(t) = S(t+1) + 1 - S(t+1)[q + S(t+1)]^{-1} S(t+1)$$

$$S(N) = 1$$

$$\min E\,l = m^2 S(1) + S(1)\sigma^2$$

b) $r_1 \neq 0 \quad r_2 = 0 \Rightarrow$ *Fullständig tillståndskännedom*

$\Rightarrow$ *Samma styrstrategi som i a)*

$$\min E\,l = m^2 S(1) + S(1)\sigma^2 + \sum_{\nu=1}^{N-1} r_1 S(\nu+1)$$

c) $r_1 \neq 0 \quad r_2 \neq 0 \Rightarrow$ *Ofullständig tillståndskännedom.*

$$u(t) = -L(t)\hat{x}(t)$$

$$L(t) = [q + S(t+1)]^{-1} S(t+1)$$

$$S(t) = S(t+1) + 1 - S(t+1)[q + S(t+1)]^{-1} S(t+1)$$

$$S(N) = 1$$

$$\hat{x}(t+1) = \Phi\hat{x}(t) + \Gamma u(t) + K(t)[y(t) - \Theta\hat{x}(t)]$$

$$K(t) = P(t)/[P(t) + r_2]$$

$$P(t+1) = P(t) + r_1 - P(t)[P(t) + r_2]^{-1} P(t)$$

$$P(1) = \sigma^2$$

$$\min E\,l = m^2 S(1) + S(1)\sigma^2 + \sum_{\nu=1}^{N-1} r_1 S(\nu+1) + \sum_{\nu=1}^{N-1} P(\nu)L(\nu)S(\nu+1)$$

8.5.2. $N \to \infty$

a) $S(t) = S(t) + 1 - S^2(t)\left[q + S(t)\right]^{-1}$

$$S^2(t) - S(t) - q = 0 \quad \Rightarrow S(t) = \frac{1}{2} (\pm) \sqrt{\frac{1}{4} + q}$$

$$\underline{L(t)} = \frac{S(t)}{q + S(t)} = \underline{\frac{0.5 + \sqrt{0.25 + q}}{0.5 + q + \sqrt{0.25+q}}}$$

b) Som a)

c) $u(t) = -L(t)\,\hat{x}(t)$

$$\hat{x}(t+1) = \hat{x}(t) + u(t) + K(t)\left[y(t) - \hat{x}(t)\right]$$

$$\hat{x} = \frac{1}{K - 1 + q} u(t) + \frac{K}{K - 1 + q} y(t)$$

$$u(t) = -\frac{L}{K - 1 + q} u(t) - \frac{LK}{K - 1 + q} y(t)$$

$$\underline{u(t)} = -\frac{LK}{K - 1 + q} \cdot \frac{K - 1 + q}{K - 1 + q + L} y(t) = \underline{-\frac{LK}{K + L - 1 + q} y(t)}$$

$$\min E\,\ell = \underbrace{m^2 S(1) + S(1)\sigma^2}_{\text{Störningar i begynnelse-tillståndet}} + \underbrace{\sum_{\nu=1}^{N-1} r_1 S(\nu+1)}_{\text{Störningar som påverkar systemet}} + \underbrace{\sum_{\gamma=1}^{N-1} P(\gamma) \frac{S^2(\gamma+1)}{q + S(\gamma+1)}}_{\text{Störningar som påverkar mätningen.}}$$

(Här finns även bidrag från störningar som påverkar systemet och begynnelsetillståndet.)

8.5.3 $x(t+1) = \Phi x(t) + e(t)$

$cov[e(t), e^T(t)] = R(t)$

$E x_0 = m$

$cov[x_0, x_0^T] = R_0$

$u(t) = -L(t)\hat{x}(t) = 0 \Rightarrow L(t) = 0$

$L(t) = [Q_2 + \Gamma^T S(t+1)\Gamma]^{-1}\Gamma^T S(t+1)\Phi = 0 \Rightarrow \Gamma = 0$

$S(t) = \Phi^T S(t+1)\Phi + Q_1$

$S(N) = Q_0$

$$\ell = x^T(N) Q_0 x(N) + \sum_{t=t_0}^{N-1} [x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t)] =$$

$$= x^T(N) Q_0 x(N) + \sum_{t=t_0}^{N-1} x^T(t) Q_1 x(t)$$

Enl. theorem 5.1

$$E\ell = m^T S(t_0) m + \mathrm{tr}\, S(t_0) R_0 + \sum_{t=t_0}^{N-1} \mathrm{tr}\, S(t+1) R(t)$$

8.5.4. $x(t+1) = \phi x(t) + \Gamma u(t) + v(t)$

$$E\ell = E\left[\sum_{s=t_0}^{t-1} x^T(s)Q_1x(s) + u^T(s)Q_2u(s)\right] + E\left[x^T(N)Q_0x(N) + \sum_{s=t}^{N-1} x^T(s)Q_1x(s) + u^T(s)Q_2u(s)\right]$$

$$V(x(t-1),t) = \min E\left[x^T(N)Q_0x(N) + \sum_{k=t}^{N-1} x^T(k)Q_1x(k) + u^T(k)Q_2u(k) \,\middle|\, x(t-1)\right]$$

<u>Bellman</u>: $V(x(t-1),t) = \min E\left[x^T(t)Q_1x(t) + u^T(t)Q_2u(t) + V(x(t),t+1) \,\middle|\, x(t-1)\right] =$

$$= \min\left\{E\left[x^T(t)Q_1x(t) \,|\, x(t-1)\right] + u^T(t)Q_2u(t) + E\left[V(x(t),t+1)\,|\,x(t-1)\right]\right\}$$

$$\begin{cases} \hat{x}(t) = E[x(t)\,|\,x(t-1)] = \phi x(t-1) + \Gamma u(t-1) \\ P(t) = \mathrm{cov}[x(t), x^T(t)\,|\,x(t-1)] = E\left[(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T \,|\, x(t-1)\right] = \\ \quad = E[\tilde{x}(t)\tilde{x}^T(t)\,|\,x(t-1)] \end{cases}$$

$$\Rightarrow V(x(t-1),t) = \min\left\{\overbrace{[\phi x(t-1) + \Gamma u(t-1)]}^{\hat{x}(t)} Q_1 \overbrace{[\phi x(t-1) + \Gamma u(t-1)]^T}^{\hat{x}^T(t)} + \mathrm{tr}\, Q_1 P(t) + \right.$$

$$\left. + u^T(t)Q_2u(t) + E\left[V(x(t),t+1)\,|\,x(t-1)\right]\right\}$$

$$V(x(N-1),N) = E\{x^T(N)Q_0x(N)\} = \hat{x}^T(N)Q_0\hat{x}(N) + \mathrm{tr}\, Q_0P(N)$$

Antag, att $V(x(t-1),t) = \hat{x}^T(t)S(t)\hat{x}(t) + s(t)$!

$$\Rightarrow V(x(t),t+1) = \hat{x}^T(t+1)S(t+1)\hat{x}(t+1) + s(t+1) =$$

$$= [\phi x(t) + \Gamma u(t)]^T S(t+1)[\phi x(t) + \Gamma u(t)] + s(t+1)$$

$$E\left[V(x(t),t+1)\,|\,x(t-1)\right] = [\phi\hat{x}(t) + \Gamma u(t)]^T S(t+1)[\phi\hat{x}(t) + \Gamma u(t)] +$$

$$+ \mathrm{tr}\,\phi^T S(t+1)\phi P(t) + s(t+1)$$

$$V(x(t-1),t) = \min\left\{\hat{x}(t)Q_1\hat{x}^T(t) + \mathrm{tr}\, Q_1P(t) + u^TQ_2u(t) + \right.$$

$$\left. + [\phi\hat{x}(t) + \Gamma u(t)]^T S(t+1)[\phi\hat{x}(t) + \Gamma u(t)] + \mathrm{tr}\,\phi^T S(t+1)\phi P(t) + s(t+1)\right.$$

(Ent. sid 272)

$$V(x(t-1),t) = \min\left\{ \hat{x}^T(t)\left[Q_1 + \Phi^T S(t+1)\Phi - L^T(Q_2 + \Gamma^T S(t+1)\Gamma)L\right]\hat{x}(t) + hQ_1 P(t) + \right.$$

$$\left. + h\Phi^T S(t+1)\Phi P(t) + s(t+1) + (u(t) + L\hat{x}(t))^T\left[Q_2 + \Gamma^T S(t+1)\Gamma\right](u(t) + L\hat{x}(t$$

$$L(t) = \left[Q_2 + \Gamma^T S(t+1)\Gamma\right]^{-1}\Gamma^T S(t+1)\Phi$$

$\Rightarrow$ Välj $u(t) = -L(t)\hat{x}(t) = -L(t)\Phi x(t-1) - L(t)\Gamma u(t-1)$

$$V(x(t-1),t) = \hat{x}^T(t)\left[Q_1 + \Phi^T S(t+1)\Phi - L^T(Q_2 + \Gamma^T S(t+1)\Gamma)L\right]\hat{x}(t) + hQ_1 P(t) +$$

$$+ h\Phi^T S(t+1)\Phi P(t) + s(t+1)$$

$$V(x(N-1),N) = \hat{x}^T(N)Q_0\hat{x}(N) + hQ_0 P(N) = \hat{x}^T(N)S(N)\hat{x}(N) + s(N)$$

$$\Rightarrow \begin{cases} S(N) = Q_0 \\ s(N) = hQ_0 P(N) \end{cases}$$

$$\min E\ell = EV(x(t_0-1),t_0) = E\left[\hat{x}^T(t_0)S(t_0)\hat{x}(t_0) + s(t_0)\right] = m^T S(t_0)m + s(t_0)$$

$$s(N) = hQ_0 P(N)$$

$$s(N-1) = hQ_1 P(N-1) + h\Phi^T S(N)\Phi P(N-1) + s(N)$$

$$\vdots$$

$$s(t_0) = hQ_0 P(N) + \sum_{\nu=t_0}^{N-1} hQ_1 P(\nu) + h\Phi^T S(\nu+1)\Phi P(\nu)$$

Theorem 4.1 : $P(t+1) = \Phi P(t)\Phi^T + R_1$

$$P(t_0) = R_0$$

$$P(t) = \Phi^{t-t_0} R_0 \Phi^{T^{t-t_0}} + \sum_{n=t_0}^{t-1} \Phi^{s-t_0} R_1 \Phi^{T^{s-t_0}}$$

Enl. sid 272

$$S(t) = \Phi^T S(t+1) \Phi + Q_1 - L^T(t)\left[Q_2 + \Gamma^T S(t+1) \Gamma\right] L(t)$$

etc.

8.6.1. $\begin{cases} x(t+1) = ax(t) + u(t-k) + v(t) \\ y(t) = x(t) + e(t) \end{cases}$

Enl. lemma 6.1.

$$\ell = x^T(N) Q_0 x(N) + \sum_{t=t_0}^{N-1} x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t) =$$

$$= x^T(t_0) S(t_0) x(t_0) + \sum_{t=t_0}^{N-1} [u(t) + L(t)x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2][u(t) + L(t)x(t)] +$$

$$+ \sum_{t=t_0}^{N-1} \{v^T(t) S(t+1)[\Phi x(t) + \Gamma u(t)] + [\Phi x(t) + \Gamma u(t)]^T S(t+1) v(t) + v^T(t) S(t+1) v(t)\} =$$

$$= x^T(t_0) S(t_0) x(t_0) + \sum_{t=t_0}^{N-1} \{v^T(t) S(t+1)[\Phi x(t) + \Gamma u(t)] + [\Phi x(t) + \Gamma u(t)]^T S(t+1) v(t) + v^T(t) S(t+1) v(t)\} +$$

$$+ \sum_{t=t_0}^{N-1} [u(t-k) + L(t) x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2][u(t-k) + L(t)x(t)] - \sum_{t=t_0-k}^{t_0-1} u^T(t) Q_2 u(t)$$

$$u(N-k) := u(N-k+1) = \ldots = u(N-1) = 0$$

$$E\ell = m^T S(t_0) m + S(t_0)\sigma^2 + \sum_{t=t_0}^{N-1} S(t+1) r_1 + E\Big\{\sum_{t_0}^{N-1} [u(t-k) + L(t)x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2]$$

$$\cdot [u(t-k) + L(t)x(t)] - \sum_{t_0-k}^{t_0-1} u^T(t) Q_2 u(t)\Big\}$$

Välj $u(t-k) = -L(t)\hat{x}(t|t-k)$

$P(t) = \mathrm{cov}[x(t) \mid Y_{t-k}]$ $\qquad$ $\hat{x}(t) = E[x(t) \mid Y_{t-k}]$

$$\min E[u(t-k) + L(t)x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2][u(t-k) + L(t)x(t)] = E \min([\ ]^T[\ ][\ ] \mid Y_{t-k}) =$$

$$= \mathrm{tr}\, L^T(t)[\Gamma^T S(t+1)\Gamma + Q_2] L(t) P(t) + [u(t-k) + L(t)\hat{x}(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2] \cdot$$

$$[u(t-k) + L(t)\hat{x}(t)]$$

$$u(t) = -L(t+k)\hat{x}(t+k|t) = -L(t+k)\Big[a^{k-1}\hat{x}(t-k) + \sum_{\nu=}^{k-1} a^{k-1-\nu} u(t-k-\nu)\Big)$$

8.6.2. $u(t) = f(Y_t)$

Enl. lemma 6.1.

$$\ell = x^T(t_0)S(t_0)x(t_0) + \sum_{t=t_0}^{N-1}[u(t)+L(t)x(t)]^T[\Gamma^T S(t+1)\Gamma + Q_2][u(t)+L(t)x(t)] +$$

$$+\sum_{t=t_0}^{N-1}\{v^T(t)S(t+1)[\Phi x(t)+\Gamma u(t)] + [\Phi x(t)+\Gamma u(t)]^T S(t+1)v(t) + v^T(t)S(t+1)v(t)\}$$

$$E\ell = m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \sum_{t=t_0}^{N-1}\mathrm{tr}\, S(t+1)R_1 + E\sum[\quad]^T[\quad][\quad]$$

$$\min E\sum_{t=t_0}^{N-1}[u(t)+L(t)x(t)]^T[\Gamma^T S(t+1)\Gamma + Q_2][u(t)+L(t)x(t)] = [\text{lemma } 3.2] =$$

$$= E\min E\left\{\sum_{t_0}^{N-1}[u(t)+L(t)x(t)]^T[\Gamma^T S(t+1)\Gamma+Q_2][u(t)+L(t)x(t)]\,\Big|\, Y_t\right\}$$

Inför $\hat{x}(t) = E[x(t)|Y_t]$ och

$P(t) = \mathrm{cov}[x(t)|Y_t]$

$$\Rightarrow \quad E\,[u(t)+L(t)x(t)]^T[\Gamma S(t+1)+Q_2][u(t)+L(t)x(t)]\,|\,Y_t\} =$$

$$=[u(t)+L(t)\hat{x}(t)]^T[\Gamma S(t+1)+Q_2][u(t)+L(t)\hat{x}(t)] + \mathrm{tr}\, L^T(t)[\Gamma S(t+1)+Q_2]L(t)P(t)$$

Minimum då $u(t) = -L(t)\hat{x}(t)$

8.6.3. $y(t)+ay(t-1)=u(t-1)+bu(t-2)+e(t)+ce(t-1)$

$$\begin{cases} x(t+1)=\overset{\Phi}{\begin{bmatrix}-a & 1\\ 0 & 0\end{bmatrix}}x(t)+\overset{\Gamma}{\begin{bmatrix}1\\ b\end{bmatrix}}u(t)+\overset{\Lambda}{\begin{bmatrix}1\\ c\end{bmatrix}}e(t+1)\\ y(t)=\overset{\Theta}{[1 \quad 0]}\,x(t)\end{cases}$$

$$E\,\ell = Ex_1^2(t+1)=Ex^T(t+1)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}x(t+1)=Ex^T(t+1)Q\,x(t+1)$$

$$E\,\ell=[\Phi x(t)+\Gamma u(t)+\Lambda e(t+1)]^TQ\,[\Phi x(t)+\Gamma u(t)+\Lambda e(t+1)]=E(x^T(t)\Phi^TQ\Phi x(t)+$$

$$+u^T(t)\Gamma^TQ\Phi x(t)+x^T(t)\Phi^TQ\Gamma u(t)+u^T(t)\Gamma^TQ\Gamma u(t))+h\,\Lambda^TQ\Lambda\,R_1(t)=$$

$$=E\Big[[u(t)+L(t)x(t)]^T\underbrace{[\Gamma^TQ\Gamma]}_{=1}[u(t)+L(t)x(t)]-x^T(t)L^T(t)L(t)x(t)+x^T\Phi^TQ\Phi x(t)\Big]$$

$$+\ h\,\Lambda^TQ\Lambda\,R_1(t)$$

$$\text{där } L(t)=\overbrace{(\Gamma^TQ\Gamma)^{-1}}\Gamma^TQ\Phi=[1 \quad b]\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-a & 1\\ 0 & 0\end{bmatrix}=[1 \quad 0]\begin{bmatrix}-a & 1\\ 0 & 0\end{bmatrix}=[-a \quad 1]$$

Välj $u(t)=-L(t)\,x(t)=ax_1(t)-x_2(t)$ gäi ty

$$\underline{x_1(t)=y(t)}=-ax_1(t-1)+x_2(t-1)+u(t-1)+e(t)=-ax_1(t-1)+x_2(t-1)+ax_1(t-1)-x_2(t-1)$$

$$+e(t)=\underline{e(t)}$$

$$\underline{x_2(t)}=bu(t-1)+ce(t)=\underline{bu(t-1)+cy(t)}$$

$$\underline{u(t)}=ay(t)-bu(t-1)-cy(t)=\underline{(a-c)y(t)-bu(t-1)}$$

$$\underline{E\ell}=h\,\Lambda^TQ\Lambda\,R_1(t)=\underline{1}$$

($|b|\geq 1$, tänk på initialvärdet $u(0)$!)

(jfr minimalvariansstrategi!)

8.6.5. Identiskt med 8.5.3

8.6.6. $x(t) = \Phi x(t) + \Gamma u(t) + v(t)$

Enligt lemma 6.1:

$$\ell = x^T(t_0) S(t_0) x(t_0) + \sum_{t=t_0}^{N-1} [u(t) + L(t)x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2][u(t) + L(t)x(t)] + \sum_{t_0}^{N-1} \{ v^T S(t+1)[\Phi x + \Gamma u] + [\Phi x + \Gamma u]^T S(t+1) v + v^T S(t+1) v \}$$

$$E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + E\left\{ \sum_{t=t_0}^{N-1} [u(t) + L(t)x(t)]^T [\Gamma^T S(t+1)\Gamma + Q_2][u(t) + L(t)x(t)] \right\} + \operatorname{tr} S(t+1) R_1$$

Ingen betingad sannolikhet

$\Rightarrow$ Välj $u(t) = -L(t)\, Ex(t) = -L(t) m(t)$

$$\begin{cases} m(t_0) = m_0 \\ m(t+1) = \Phi m(t) + \Gamma u(t) = (\Phi - \Gamma L) m(t) \end{cases}$$

$$E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} L^T(t) [\Gamma^T S(t+1) \Gamma + Q_2] L(t) R(t) + \operatorname{tr} S(t+1) R_1$$

$R(t) = \operatorname{cov}[x(t), x^T(t)]$

$R(t_0) = R_0$

$$R(t+1) = E[(x(t+1) - m(t+1))(x(t+1) - m(t+1))^T] =$$

$$= E[(\Phi x(t) - \Gamma L m(t) + v(t) - \Phi m(t) + \Gamma L m(t))(\Phi x(t) - \Gamma L m(t) + v(t) - \Phi m(t) + \Gamma L m(t))^T]$$

$$= E[\Phi (x(t) - m(t))(x(t) - m(t))^T \Phi^T] + R_1(t) = \Phi R(t) \Phi^T + R_1(t)$$

$$L^T(t) [\Gamma^T S(t+1) \Gamma + Q_2] L(t) = Q_1 + \Phi^T S(t+1) \Phi - S(t)$$

$$E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} Q_1 R(t) + \operatorname{tr} \Phi^T S(t+1) \Phi R(t) - \operatorname{tr} S(t) R(t) + \operatorname{tr} S(t+1) R_1(t)$$

$$= m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} Q_1 R(t) + \operatorname{tr}[S(t+1) \Phi R(t) \Phi^T - S(t) R(t)] + \operatorname{tr} S(t+1) R_1(t) =$$

$$= m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} Q_1 R(t) + \operatorname{tr}[S(t+1) R(t+1) - S(t) R(t) - \overbrace{S(t+1) R_1(t)}^{(+\operatorname{tr} S(t+1) R_1)}] =$$

$$= m^T S(t_0) m + \operatorname{tr} Q_0 R(N) + \sum_{t=t_0}^{N-1} \operatorname{tr} Q_1 R(t)$$

8.6.3

$$y(t) + a\,y(t-1) = u(t-1) + b\,u(t-2) + e(t) + c\,e(t-1)$$

$$\begin{cases} x(t+1) = \overset{\Phi}{\begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix}} x(t) + \overset{\Gamma}{\begin{bmatrix} 1 \\ b \end{bmatrix}} u(t) + \overset{\Lambda}{\begin{bmatrix} 1 \\ c \end{bmatrix}} e(t+1) \\ y(t) = \underset{\theta}{[1\ 0]}\, x(t) \end{cases}$$

$$E\,l = E\,x_1^2(t+1) = E\,x^T(t+1)\begin{bmatrix}1&0\\0&0\end{bmatrix}x(t+1) = E\,x^T(t+1)\,Q\,x(t+1)$$

$$= E[\Phi x(t) + \Gamma u(t) + \Lambda e(t+1)]^T Q\,[\Phi x(t) + \Gamma u(t) + \Lambda e(t+1)]$$

$$= E[x^T(t)\Phi^T Q \Phi x(t) + u^T(t)\Gamma^T Q \Phi x(t) + x^T(t)\Phi^T Q \Gamma u(t) + u^T(t)\Gamma^T Q\Gamma u(t)] +$$

$$+ \operatorname{tr} \Lambda^T Q \Lambda R_1(t+1) = E[(u(t) + L(t)x(t))^T(\Gamma^T Q \Gamma)(u(t) + L(t)x(t))] +$$

$$+ E[x^T(t)\{\Phi^T Q \Phi - L^T(t)(\Gamma^T Q\Gamma)L(t)\}x(t)] \qquad + \operatorname{tr}\Lambda^T Q \Lambda R_1 \qquad (*)$$

där $L(t) = (\Gamma^T Q \Gamma)^{-1}\Gamma^T Q \Phi = \left([1\ b]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1\\b\end{bmatrix}\right)^{-1}[1\ b]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}-a&1\\0&0\end{bmatrix} = [-a\ \ 1]$

El minimeras för $u(t) = -L(t)\hat{x}(t|t)$ (tillåten!)

Andra termen i $(*) = 0$ ty $\Phi^T Q \Phi = \begin{bmatrix}-a\\1\end{bmatrix}[-a\ \ 1]$

$$El = \operatorname{tr} P(t|t) L^T(\Gamma^T Q \Gamma) L + \operatorname{tr}\Lambda^T Q \Lambda R_1$$

stationär Riccati ($t_0 \to -\infty$) ger* $P(t|t-1) = R_1\begin{bmatrix}1&c\\c&c^2\end{bmatrix}$ och $P(t|t) = 0$, dvs

$\hat{x}(t+1|t+1) = \Phi\hat{x}(t|t) + \Gamma u(t) + P(t+1|t+1)\theta^T(\theta P\theta^T + 0)^{-1}\cdot(y(t+1) - \theta\Phi\hat{x}(t|t) - \theta\Gamma u(t))$

$$\begin{cases}\hat{x}_1(t|t) = y(t) \\ \hat{x}_2(t|t) = b\,u(t-1) + c\,y(t)\end{cases}$$

$\hat{x}_2(t+1|t+1) = -c\,\hat{x}_2(t|t) + c\,(y(t+1) + ay(t)) + (b-c)u(t)$
$= -c\hat{x}_2(t|t) + cy(t+1) + acy(t) + bu(t) - c[ay(t) - \hat{x}$
$= cy(t+1) + bu(t)$

dvs $u(t) = (a-c)\,y(t) - b\,u(t-1)$

$$El = \operatorname{tr}\Lambda^T Q \Lambda R_1 = 1$$

jfr minimalvarians strategin

obs! instabilt om $|b| \geq 1$

$*$ $P = \begin{bmatrix}1\\0\end{bmatrix}[-a\ 1]\begin{bmatrix}0\\1\end{bmatrix}\left(p_{22} - \frac{p_{12}^2}{p_{11}}\right)[0\ 1]\begin{bmatrix}-a\\1\end{bmatrix}[1\ 0] + \begin{bmatrix}1\\c\end{bmatrix} r_1\,[1\ c]$ $\Rightarrow$ $p_{11}^2 - r_1(1+c^2)p_{11} + r_1 c^2 = 0$

$\Rightarrow$ $p_{11} = r_1 \cdot \begin{cases}1 \\ c^2\end{cases}$ $|c| < 1$

8.6.4. $\ell = \frac{1}{N}\sum_{t=0}^{N} y^2(t)$ f.ö. som i 8.6.3

Teorem 8.6.3 direkt ger: $u(t) = -L(t)\hat{x}(t|t)$ $L(t) = (\Gamma^T S \Gamma)^{-1}\Gamma^T S \phi$

och $E\ell = \frac{1}{N}\left\{ \operatorname{tr} S(0)R_0 + \sum_{t=0}^{N-1} \operatorname{tr} S(t+1)R_1 + \sum_{t=0}^{N-1} \operatorname{tr} P(t|t)\, L^T(t)\,(\Gamma^T S(t+1)\Gamma + Q_2)\, L(t) \right\}$

Undersök $N \to \infty$ Stationär Riccati ekvation

$$S = \phi^T S \phi + Q - \phi^T S \Gamma (\Gamma^T S \Gamma)^{-1} \Gamma^T S \phi$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -a \\ 1 \end{bmatrix} [1\ 0]\, S \begin{bmatrix} 1 \\ b \end{bmatrix} [-a\ 1] - \begin{bmatrix} -a \\ 1 \end{bmatrix} [1\ 0]\, S \begin{bmatrix} 1 \\ b \end{bmatrix} \left\{ [1\ b]\, S \begin{bmatrix} 1 \\ b \end{bmatrix} \right\}^{-1} [1\ b]\, S \begin{bmatrix} 1 \\ b \end{bmatrix} [-a\ 1]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -a \\ 1 \end{bmatrix} [-a\ 1] \cdot \left(s_{11} - \frac{(s_{11} + b s_{12})^2}{s_{11} + 2b s_{12} + b^2 s_{22}} \right)$$

dvs
$$\begin{cases} s_{11} = 1 + a^2 \cdot f(s) \\ s_{12} = -a\, f(s) \\ s_{22} = f(s) \end{cases}, \quad f(s) = \frac{s_{11}^2 + \cancel{2b s_{11} s_{12}} + b^2 s_{11} s_{22} - s_{11}^2 - \cancel{2b s_{12} s_{11}} - b^2 s_{12}^2}{s_{11} + 2b s_{12} + b^2 s_{22}}$$

$$= b^2 \frac{s_{11} s_{22} - s_{12}^2}{s_{11} + 2b s_{12} + b^2 s_{22}}$$

$\det S = f(s)\{1 + a^2 f(s) - a^2 f(s)\} = f(s)$

- en lösning $f(s) = 0 \Rightarrow S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow L = [-a\ 1]$

- annan lösning $1 = b^2 \dfrac{1 + a^2 s_{22} - a^2 s_{22}}{1 + a^2 s_{22} - 2ab s_{22} + b^2 s_{22}}$ $\Rightarrow \lambda(\phi - \Gamma L) = \begin{cases} 0 \\ -b \end{cases}$

$$1 + (a-b)^2 s_{22} = b^2$$

$$\Rightarrow f(s) = s_{22} = \frac{b^2 - 1}{(a-b)^2} > 0 \quad \text{om } |b| > 1$$

$$\Rightarrow L = \frac{1 + a(a-b) f}{1 + (a-b)^2 f} [-a\ 1] = \frac{a - 1/b}{a - b} [-a\ 1]$$

$$\Rightarrow \lambda(\phi - \Gamma L) = \begin{cases} 0 \\ -1/b \end{cases}$$

men $S(N) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow S(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ \forall t$, dvs instabil om $|b| > 1$

$$\Rightarrow E\ell \approx \operatorname{tr} S(t) R_1 = \operatorname{tr} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ c & c^2 \end{bmatrix} = 1$$ samma som i 8.6.3.

den stabila regulatorn ger $E\ell = \operatorname{tr} S R_1 = [1\ c]\, S \begin{bmatrix} 1 \\ c \end{bmatrix} = 1 + a^2 f + 2ca f + c^2 f =$

$$= 1 + \frac{(a+c)^2}{(a-b)^2} (b^2 - 1) \qquad |b| > 1$$

8.6.7. $x(t+1) = \Phi x(t) + \Gamma u(t) + v(t)$

$$\ell = x^T(N) Q_0 x(N) + \sum_{t=t_0}^{N-1} \left[x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t)\right]$$

Enl. lemma 6.1.

$$\ell = x^T(t_0) S(t_0) x(t_0) + \sum_{t=t_0}^{N-1} \left[u(t) + L(t) x(t)\right]^T \left[\Gamma^T S(t+1) \Gamma + Q_2\right]\left[u(t) + L(t) x(t)\right] +$$

$$+ \sum_{t=t_0}^{N-1} \left\{ v^T(t) S(t+1)\left[\Phi x(t) + \Gamma u(t)\right] + \left[\Phi x(t) + \Gamma u(t)\right]^T S(t+1) v(t) + v^T(t) S(t+1) v(t) \right\}$$

$$E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} S(t+1) R_1(t) + E \sum_{t=t_0}^{N-1} \left[u(t) + L(t) x(t)\right]^T \left[\Gamma^T S(t+1) \Gamma + Q_2\right]\left[u(t) + L(t) x(t)\right]$$

Optimal strategi $\Rightarrow u(t_0) = 0$

$$E \sum_{t=t_0}^{N-1} \left[u(t) + L(t) x(t)\right]^T \left[\Gamma^T S(t+1) \Gamma + Q_2\right]\left[u(t) + L(t) x(t)\right] = E \sum_{t=t_0+1}^{N-1} \left[u(t) + L(t)\left[\Phi x(t-1) + \Gamma u(t-1) + v(t-1)\right]\right]^T$$

$$\left[\Gamma^T S(t+1) \Gamma + Q_2\right]\left[u(t) + L(t)\left[\Phi x(t-1) + \Gamma u(t-1) + v(t-1)\right]\right]$$

Sätt $\underline{u(t) = -L(t)\left[\Phi x(t-1) + \Gamma u(t-1)\right]}$

$$E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \sum \operatorname{tr} S(t+1) R_1(t) + \sum_{t=t_0+1}^{N-1} \operatorname{tr} L^T(t)\left[\Gamma^T S(t+1) \Gamma + Q_2\right] L(t) R_1(t-1) +$$

$$\underline{+ \operatorname{tr} L^T(t_0)\left[\Gamma^T S(t_0+1) \Gamma + Q_2\right] L(t_0) R_0}$$

8.6.8. Enligt lemma 6.1.

$$\ell = x^T(t_0)S(t_0)x(t_0) + \sum_{t=t_0}^{N-1}[u(t)+L(t)x(t)]^T[\Gamma^T S(t+1)\Gamma + Q_2][u(t)+L(t)x(t)] +$$

$$+ \sum_{t=t_0}^{N-1}\{v^T(t)S(t+1)[\Phi x(t)+\Gamma u(t)] + [\Phi x(t)+\Gamma u(t)]^T S(t+1)v(t) + v^T(t)S(t+1)v(t)\} =$$

$$= x^T(t_0)S(t_0)x(t_0)$$

$$\min E\ell = E\min E[\ell \mid Y_{t-k}] = m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \sum_{t=t_0}^{N-1}\mathrm{tr}\, S(t+1)R_1(t) +$$

$$+ E\min E\left\{\sum_{t=t_0}^{N-1}[u(t)+L(t)x(t)]^T[\Gamma^T S(t+1)\Gamma + Q_2][u(t)+L(t)x(t)] \mid Y_{t-k}\right\}$$

Inför $\hat{x}(t) = E[x(t) \mid Y_{t-k}]$

$$P(t) = \mathrm{cov}[x(t) \mid Y_{t-k}]$$

$$\min E\ell = m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \sum_{t=t_0}^{N-1}\mathrm{tr}\, S(t+1)R_1(t) +$$

$$+ \sum_{t=t_0}^{N-1}[u(t)+L(t)\hat{x}(t)]^T[\Gamma^T S(t+1)\Gamma + Q_2][u(t)+L(t)\hat{x}(t)] +$$

$$+ \sum_{t=t_0}^{N-1}\mathrm{tr}\, L^T(t)[\Gamma^T S(t+1)\Gamma + Q_2]L(t)P(t)$$

$\Rightarrow$ $\underline{\text{Välj } u(t) = -L(t)\hat{x}(t) = -L(t)E[x(t) \mid Y_{t-k}]}$

$$\underline{\min E\ell = m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \sum_{t=t_0}^{N-1}\mathrm{tr}\, S(t+1)R_1(t) + \sum_{t=t_0}^{N-1}\mathrm{tr}\, L^T(t)[\Gamma^T S(t+1)\Gamma + Q_2]L(t)P(t)}$$

8.7.1. $\begin{cases} dx = u\,dt + dv \\ dy = x\,dt + de \end{cases}$

$$\ell = \int_0^T [x^2(t) + q u^2(t)]\,dt$$

Lemma 7.1 $\Rightarrow \ell = x^2(0)S(0) + \int_0^T [u + \frac{1}{q}Sx]^2 q\,dt + \int_0^T r_1 S\,dt + 2\int_0^T S x\,dv$

$$-\frac{dS}{dt} = 1 - \frac{1}{q}S^2$$

i) Open loop

$E\ell = m^2 S(0) + S(0) r_0 + \int_0^T r_1 S\,dt$, om $u(t) = -\frac{1}{q} S m$

$-\frac{dS}{dt} = 1 - \frac{S^2}{q}$ $\quad S(T) = 0$

ii) $e \equiv 0 \Leftrightarrow$ Fullständig tillståndsinformation

$E\ell = m^2 S(0) + S(0) r_0 + \int_0^T r_1 S\,dt$, om $u(t) = -\frac{1}{q} S x$

iii) Ofullständig tillståndsinformation

$E\ell = m^2 S(0) + S(0) r_0 + \int_0^T r_1 S\,dt + \int_0^T S^2 \frac{1}{q} P\,dt$, om $u(t) = -\frac{S}{q}\hat{x}$

$P = \operatorname{cov}[x(t) \mid y_t]$

$d\hat{x} = u\,dt + K(t)[dy - \hat{x}\,dt]$

$K(t) = \frac{P(t)}{r_2}$

$\dot{P} = r_1 - P^2 \cdot \frac{1}{r_2}$

$P(0) = r_0$

8.7.3. $dx = Ax\,dt + dv$

Lemma 7.1 $\Rightarrow E\left\{x^T(t_1)Q_0x(t_1) + \int_{t_0}^{t_1} x^T(s)Q_1x(s)\,ds\right\} = E\Big\{x^T(t_0)S(t_0)x(t_0) +$

$$+\int_{t_0}^{t_1} \mathrm{tr}\, RS\,dt + \int_{t_0}^{t_1} dv^T Sx + \int_{t_0}^{t_1} x^T S\,dv\Big\} = \underline{m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \int_{t_0}^{t_1} \mathrm{tr}\, RS\,dt}$$

$$-\frac{dS}{dt} = A^T S + SA + Q_1 \qquad S(t_1) = Q_0$$

$$x^T(t_1)Q_0x(t_1) = x^T(t_1)S(t_1)x(t_1) = x^T(t_0)S(t_0)x(t_0) + \int_{t_0}^{t_1} d(x^T Sx)$$

$$d(x^T Sx) = dx^T Sx + x^T dS\,x + x^T S\,dx + \mathrm{tr}\, SR\,dt =$$

$$= x^T A^T Sx\,dt + dv^T Sx + x^T\left[-A^T S - SA - Q_1\right]x\,dt + x^T SAx\,dt + x^T S\,dv +$$

$$+ \mathrm{tr}\, SR\,dt = dv^T Sx - x^T Q_1 x\,dt + x^T S\,dv + \mathrm{tr}\, SR\,dt$$

$$\Rightarrow E\left\{x^T(t_1)Q_0x(t_1) + \int_{t_0}^{t_1} x^T(s)Q_1x(s)\,ds\right\} =$$

$$= E\left\{x^T(t_0)S(t_0)x(t_0) + \int_{t_0}^{t_1} dv^T Sx + \int_{t_0}^{t_1} x^T S\,dv + \int_{t_0}^{t_1} \mathrm{tr}\, SR\,dt\right\} =$$

$$= \underline{m^T S(t_0)m + \mathrm{tr}\, S(t_0)R_0 + \int_{t_0}^{t_1} \mathrm{tr}\, SR\,dt}$$

$\#$

8.7.6. $\begin{cases} dx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x\,dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,dt + dv \\ dy = [1 \quad 0]\, x\,dt + de \end{cases}$

$$\ell = \int_{t_0}^{t_1} [x_1^2(t) + q u^2(t)]\,dt = \int_{t_0}^{t_1} [x^T(t) Q_1 x(t) + q u^2(t)]\,dt$$

$\underline{u(t) = -L(t)\hat{x}(t)}$

$$L(t) = -\frac{1}{q}[0 \quad 1]S = -\frac{1}{q}(0 \quad 1)\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = -\frac{1}{q}(s_2 \quad s_3)$$

$$\hat{x}(t) = E[x(t) \mid \mathcal{Y}_t]$$

$$-\frac{dS}{dt} = A^T S + SA + Q_1 - SB\frac{1}{q}B^T S^T$$

Stationär: $A^T S + SA + Q_1 - SB\frac{1}{q}B^T S^T = 0$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{q}[0 \quad 1]\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix} + \begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_2 \\ s_3 \end{bmatrix}[s_2 \quad s_3]\cdot\frac{1}{q} = 0$$

$$\begin{pmatrix} 1 - s_2^2\cdot\frac{1}{q} & s_1 - s_2 s_3\cdot\frac{1}{q} \\ s_1 - s_2 s_3\cdot\frac{1}{q} & 2s_2 - s_3^2\cdot\frac{1}{q} \end{pmatrix} = 0$$

$$s_2 = \sqrt{q}$$

$$2\sqrt{q} - s_3^2\cdot\frac{1}{q} = 0 \Rightarrow s_3 = \sqrt{2}\, q^{\frac{3}{4}}$$

$$s_1 = \sqrt{q}\sqrt{2}\, q^{\frac{3}{4}}\cdot\frac{1}{q} = \sqrt{2}\, q^{\frac{1}{4}}$$

$$L(t) = -\frac{1}{q}(s_2 \quad s_3) = \underline{-\left(\frac{1}{\sqrt{q}} \quad \sqrt{2}\, q^{-\frac{3}{4}}\right)}$$

$$\underline{\min E\ell = m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 + \int_{t_0}^{t_1} \operatorname{tr} S R_1\,dt + \int_{t_0}^{t_1} \operatorname{tr} S B\frac{1}{q}B^T S P\,dt}$$