7.2.1. $y(t) = s(t) + n(t)$

$$\hat{s}(t+h) = \int_{-\infty}^{t} g(t-\tau)\, y(\tau)\, d\tau = \int_{0}^{\infty} g(u)\, y(t-u)\, du$$

$$E\left[s(t+h) - \hat{s}(t+h)\right]^2 = E\left[s^2(t+h) - 2s(t+h)\hat{s}(t+h) + \hat{s}^2(t+h)\right] =$$

$$= r_s(0) - E\, 2 s(t+h) \int_0^\infty g(u)\left[s(t-u) + n(t-u)\right] du \; +$$

$$+ E \int_0^\infty g(u)\, y(t-u)\, du \int_0^\infty g(v)\, y(t-v)\, dv = r_s(0) - \underbrace{2\int_0^\infty g(u)\, r_s(h+u)\, du}$$

$$\underbrace{+ \int_0^\infty g(u)\, du \int_0^\infty g(v)\left[r_s(u-v) + r_n(u-v)\right] dv}$$

7.2.2. $J(g) = r_s(0) - 2\int_0^\infty g(u)\, r_s(u+h)\, du + \int_0^\infty\!\!\int_0^\infty g(u)\left[r_s(u-v) + r_n(u-v)\right] \cdot$

$\cdot\, g(v)\, du\, dv$

$$\underbrace{J(g+\delta g)} = r_s(0) - 2\int_0^\infty g(u)\, r_s(u+h)\, du - 2\int_0^\infty \delta g(u)\, r_s(u+h)\, du \; +$$

$$+ \int_0^\infty\!\!\int_0^\infty g(u)\left[r_s(u-v) + r_n(u-v)\right] g(v)\, du\, dv \; +$$

$$+ \int_0^\infty\!\!\int_0^\infty \delta g(u)\left[r_s(u-v) + r_n(u-v)\right] g(v)\, du\, dv + \int_0^\infty\!\!\int_0^\infty g(u)\left[r_s(u-v) + \right.$$

$$\left. + r_n(u-v)\right] \delta g(v)\, du\, dv + \int_0^\infty\!\!\int_0^\infty \delta g(u)\left[r_s(u-v) + r_n(u-v)\right] \delta g(v)\, du\, dv$$

$$= J(g) - 2\int_0^\infty \delta g(u)\left[r_s(u+h) - \int_0^\infty \left(r_s(u-v) + r_n(u-v)\right) g(v)\, dv\right] du \; +$$

$$+ \int_0^\infty\!\!\int_0^\infty \delta g(u)\left[r_s(u-v) + r_n(u-v)\right] \delta g(v)\, du\, dv = \underbrace{J(g) + J_1 + J_2}$$

#

$$J(g+\delta g) = J(g) + J_1 + J_2$$

$$J(g+\delta g) \geq J(g) \quad \Leftrightarrow \quad \underline{J_1 = 0}$$

7.2.3. $$y(t) = \sum_{n=t_0}^{t} g(t,n)e(n)$$

$$y(t+k) - \hat{y}(t+k|t) = \sum_{n=t_0}^{t+k} g(t+k,n)\,e(n) - \hat{y}(t+k|t) =$$

$$= \underbrace{\sum_{n=t_0}^{t} g(t+k,n)e(n)}_{\text{Gå att bestämma}} + \underbrace{\sum_{n=t+1}^{t+k} g(t+k,n)e(n)}_{\text{Gå ej att bestämma}} - \hat{y}(t+k|t)$$

Välj $\hat{y}(t+k|t) = \sum_{n=t_0}^{t} g(t+k,n)e(n) = \sum_{n=t_0}^{t} h(t,n)y(n)$

$$y(t_0) = g(t_0,t_0)e(t_0) \Rightarrow e(t_0) = \frac{1}{g(t_0,t_0)}y(t_0)$$

$y(t_0+1) = \dfrac{g(t_0+1,t_0)}{g(t_0,t_0)}y(t_0) + g(t_0+1,t_0+1)e(t_0+1)$ ger $e(t_0+1)$

<u>etc</u>.

7.2.4. $y(t) = \int_{t_0}^{t} g(t,s)\,dw(s)$

$$E\left[y(t+h) - \hat{y}(t+h|t)\right]^2 = E\left[\int_{t_0}^{t} g(t+h,s)\,dw(s) + \int_{t}^{t+h} g(t+h,s)\,dw(s) - \hat{y}(t+h|t)\right]^2 =$$

$$= E\left(\int_{t_0}^{t} g(t+h,s)\,dw(s)\right)^2 + E\left(\int_{t}^{t+h} g(t+h,s)\,dw(s)\right)^2 + E\hat{y}^2(t+h|t) - 2E\hat{y}(t+h|t)\cdot$$

$$\cdot\int_{t_0}^{t} g(t+h,s)\,dw(s) \geq E\left(\int_{t}^{t+h} g(t+h,s)\,dw(s)\right)^2$$

Likhet då $\left(\int_{t_0}^{t} g(t+h,s)\,dw(s)\right)^2 + \hat{y}^2(t+h|t) - 2\hat{y}(t+h|t)\int_{t_0}^{t} g(t+h,s)\,dw(s) =$

$$\Rightarrow \hat{y}(t+h|t) = \int_{t_0}^{t} g(t+h,s)\,dw(s)$$

Prediktionsfelet $= \int_{t}^{t+h} g(t+h,s)\,dw(s)$

7.2.5. $y(t) = \int_{t_0}^{t} g(t,s)\,dw(s)$ $\qquad \hat{y}(t+h|t) = \alpha y(t)$

$$E\left[y(t+h) - \hat{y}(t+h|t)\right]^2 = E\left[\int_{t_0}^{t} g(t+h,s)\,dw(s) + \int_{t}^{t+h} g(t+h,s)\,dw(s) - \alpha\int_{t_0}^{t} g(t,s)\,dw\right.$$

$$= E\left[\int_{t_0}^{t}\left(g(t+h,s) - \alpha g(t,s)\right)dw(s) + \int_{t}^{t+h} g(t+h,s)\,dw(s)\right]^2$$

Välj α så att $\left(g(t+h,s) - \alpha g(t,s)\right)^2$ minimeras.

<u>Spec</u>: $g(t,s) = (t-s)\,e^{-(t-s)}$, $t_0 = -\infty$

$$\left(g(t+h,s) - \alpha g(t,s)\right)^2 = \eta$$

$$(t+h-s)^2 e^{-2(t+h-s)} + \alpha^2 (t-s)^2 e^{-2(t-s)} \quad - 2\alpha (t+h-s)(t-s) e^{-(2t+h-2s)} = \eta$$

$$\alpha = K e^{-h}$$

$$\left[(t+h-s)^2 + K^2 (t-s)^2 - 2K(t+h-s)(t-s)\right] e^{-2(t+h-s)} = \eta$$

$$\left[(t+h-s) - K(t-s)\right]^2 e^{-2(t+h-s)} = \eta$$

$K = 1$ minimizes η.

$$\hat{y}(t+h/t) = \alpha\, y(t) = e^{-h} y(t)$$

7.3.1. $x \in N(a, \sigma_0)$

$v \in N(0, \sigma)$

$y = x + v$

$E(x|y) = m_x + R_{xy} R_y^{-1}(y - m_y)$

$m_x = a$

$R_{xy} = E(x - m_x)(y - m_y) = E(x-a)(x-a+v) = E(x-a)(x-a) +$

$+ E(x-a)v = \sigma_0^2$

$R_y = E(y - m_y)(y - m_y) = E(x-a+v)(x-a+v) = E(x-a)^2 + Ev^2 =$

$= \sigma_0^2 + \sigma^2$

$$\hat{x} = E(x|y) = a + \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2}(y - a) = \frac{\sigma^2}{\sigma_0^2 + \sigma^2} a + \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2} y$$

7.3.2. $x \in N(a, \sigma_0)$

$v \in N(0, \sigma_1)$

$e \in N(0, \sigma_2)$

$y = x + v + e$

$E(x|y) = m_x + R_{xy} R_y^{-1}(y - m_y)$

$R_{xy} = E(x-a)(x-a+v+e) = E(x-a)(x-a) = \sigma_0^2$

$R_y = E(x-a+v+e)(x-a+v+e) = \sigma_0^2 + \sigma_1^2 + \sigma_2^2$

$$\hat{x} = E(x|y) = a + \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2 + \sigma_2^2}(y - a) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_0^2 + \sigma_1^2 + \sigma_2^2} a + \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2 + \sigma_2^2} y$$

$$\lim_{\sigma_0 \to \infty} \hat{x} = y$$

$$E(\tilde{x}\tilde{x}^T | y) = R_x - R_{xy} R_y^{-1} R_{yx} = \sigma_0^2 - \sigma_0^2 \frac{1}{\sigma_0^2 + \sigma_1^2 + \sigma_2^2} \sigma_0^2 =$$

$$= \frac{\sigma_0^2(\sigma_1^2 + \sigma_2^2)}{\sigma_0^2 + \sigma_1^2 + \sigma_2^2}$$

7.3.3. $Ex = m_x$

$$E(x - m_x)(x - m_x)^T = R_x$$

$$y = Cx + e$$

$$Ee = 0$$

$$Eee^T = R_e$$

$$E(x|y) = m_x + R_{xy} R_y^{-1}(y - m_y)$$

$$R_{xy} = E(x - m_x)(Cx + e - Cm_x)^T = E(x - m_x)(x - m_x)^T C^T = R_x C^T$$

$$R_y = E(Cx - Cm_x + e)(Cx - Cm_x + e)^T = CR_x C^T + R_e$$

$$\hat{x} = E(x|y) = m_x + R_x C^T (CR_x C^T + R_e)^{-1}(y - Cm_x)$$

$$E(\tilde{x}\tilde{x}^T | y) = R_x - R_{xy} R_y^{-1} R_{yx} = R_x - R_x C^T (CR_x C^T + R_e)^{-1} C R_x$$

7.4.1.
$$\begin{cases} x(t+1) = a x(t) + v(t) \\ y(t) = x(t) + e(t) \end{cases}$$

$R_1 = 1 \qquad R_2 = \sigma^2 \qquad m = 1 \qquad R_0 = \sigma_0^2 \qquad a = 1$

$$\begin{cases} \hat{x}(t+1) = \hat{x}(t) + K(t)[y(t) - \hat{x}(t)] \\ \hat{x}(t_0) = 1 \end{cases}$$

$$K(t) = P(t)\left[P(t) + \sigma^2\right]^{-1}$$

$$\begin{cases} P(t+1) = P(t) + 1 - P(t)[P(t) + \sigma^2]^{-1} P(t) \\ P(t_0) = \sigma_0^2 \end{cases}$$

<u>Stationär lösning</u>

$$P(t+1) = P(t) = P(t) + 1 - \frac{P(t)^2}{P(t) + \sigma^2}$$

$$P^2 - P - \sigma^2 = 0$$

$$P = +\frac{1}{2} \overset{+}{(-)} \sqrt{\frac{1}{4} + \sigma^2}$$

$$\underline{K} = \frac{+\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma^2}}{+\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma^2} + \sigma^2} = \frac{\sigma^2}{\left(\sqrt{\frac{1}{4} + \sigma^2} + \frac{1}{2} + \sigma^2\right)\left(\sqrt{\frac{1}{4} + \sigma^2} - \frac{1}{2}\right)} =$$

$$= \frac{\sigma^2}{\sigma^2 + \sigma^2\left(\sqrt{\frac{1}{4} + \sigma^2} - \frac{1}{2}\right)} = \underline{\underline{\frac{1}{\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma^2}}}}$$

7.4.2. $\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Theta = (1 \quad 0) \quad R_1 = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \quad R_2 = 1$

<u>Stationärt</u>

$$P = \Phi P \Phi^T + R_1 - \Phi P \Theta^T \left[\Theta P \Theta^T + R_2\right]^{-1} \Theta P \Phi^T =$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left[(1 \quad 0)\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\right]^{-1} \cdot$$

$$\cdot (1 \quad 0)\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} p_1+p_2 & p_2+p_3 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\frac{1}{p_1+1}(p_1 \quad p_2)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} p_1+2p_2+p_3+r_1 & p_2+p_3 \\ p_2+p_3 & p_3+r_2 \end{pmatrix} - \frac{1}{p_1+1}\begin{pmatrix} p_1+p_2 \\ p_2 \end{pmatrix}\left[p_1+p_2 \quad p_2\right] =$$

$$= \begin{pmatrix} p_1+2p_2+p_3+r_1 & p_2+p_3 \\ p_2+p_3 & p_3+r_2 \end{pmatrix} - \frac{1}{p_1+1}\begin{pmatrix} (p_1+p_2)^2 & p_2(p_1+p_2) \\ p_2(p_1+p_2) & p_2^2 \end{pmatrix}$$

$$\begin{cases} p_1 = p_1 + 2p_2 + p_3 + r_1 - \dfrac{(p_1+p_2)^2}{1+p_1} \\ p_2 = p_2 + p_3 - p_2(p_1+p_2)/p_1+1 \\ p_3 = p_3 + r_2 - p_2^2/p_1+1 \end{cases} \Rightarrow \begin{cases} 0 = (p_1+1)(2p_2+p_3+r_1) - (p_1+p_2)^2 \\ 0 = p_3(p_1+1) - p_2(p_1+p_2) \\ 0 = r_2(p_1+1) - p_2^2 \end{cases}$$

$$\begin{cases} p_1p_3 + p_1r_1 + 2p_2 + p_3 + r_1 - p_1^2 - p_2^2 = 0 & (1) \\ p_1p_3 + p_3 - p_1p_2 - p_2^2 = 0 & (2) \\ r_2p_1 + r_2 - p_2^2 = 0 & (3) \end{cases}$$

(Härur löses lätt P.

$$K = \Phi P \Theta^T \left[\Theta P \Theta^T + R_2\right]^{-1} = \underline{\frac{1}{p_1+1}\begin{pmatrix} p_1+p_2 \\ p_2 \end{pmatrix}}$$

7.4.3. $\begin{cases} x(t+1) = \Phi x(t) + \Gamma u(t) + v(t) \\ y(t) = \Theta x(t) + e(t) \end{cases}$

$E v(t) v^T(s) = \delta_{s,t} R_1$

$E v(t) e^T(s) = \delta_{s,t} R_{12}$

$E e(t) e^T(s) = \delta_{s,t} R_2$

$E x_0 x_0^T = R_0$

$\hat{x}(t+1) = E[x(t+1) | \mathcal{Y}_{t-1}] + E[x(t+1) | \tilde{y}(t)] - E x(t+1)$

1) $E[x(t+1)|\mathcal{Y}_{t-1}] = E[\Phi x(t) + \Gamma u(t) + v(t) | \mathcal{Y}_{t-1}] = \Phi E[x(t)|\mathcal{Y}_{t-1}] + \Gamma E[u(t)|\mathcal{Y}_{t-1}] +$

$+ E[v(t) | \mathcal{Y}_{t-1}] = \Phi \hat{x}(t) + \Gamma u(t)$

2) Theorem 3.2 : $E[x|y] = m_x + R_{xy} R_y^{-1} (\tilde{y} - m_{\tilde{y}})$

$R_{x\tilde{y}} = \operatorname{cov}(x(t+1), \tilde{y}(t)) = E[\Phi x(t) + \Gamma u(t) + v(t) - \Phi E x(t) - \Gamma E u(t)][\Theta \tilde{x}(t) + e(t)]^T$

$= E[\Phi \tilde{x} + \Phi \hat{x} + v][\Theta \tilde{x} + e]^T = \Phi P(t) \Theta^T + R_{12}$ ($\hat{x}$ och $\tilde{x}$ är oberoende!)

$P(t) = E \tilde{x} \tilde{x}^T$

$R_{\tilde{y}} = E[\Theta \tilde{x}(t) + e(t)][\Theta \tilde{x}(t) + e(t)]^T = \Theta P(t) \Theta^T + R_2$

$E(x(t+1) | \tilde{y}(t)) = E x(t+1) + K(t) \tilde{y}(t)$

$K(t) = R_{x\tilde{y}} R_{\tilde{y}}^{-1} = [\Phi P(t) \Theta^T + R_{12}][\Theta P(t) \Theta^T + R_2]^{-1}$

$\hat{x}(t+1) = \Phi \hat{x}(t) + \Gamma u(t) + K(t)[y(t) - \Theta \hat{x}(t)]$

7.4.4. $\begin{cases} x(t+1) = \Phi x(t) + \Gamma e(t) \\ y(t) = \Theta x(t) + e(t) \end{cases}$

$Ee = 0 \qquad Ex_0 = a$

$Eee^T = R_2 \qquad Ex_0x_0^T = R_0$

Sätt $v(t) = \Gamma e(t)$.

$Evv^T = E\Gamma ee^T\Gamma^T = \Gamma R_2 \Gamma^T \equiv R_1$

$Eev^T = E\, ee^T\Gamma^T = R_2\Gamma^T \equiv R_{21}$

$Eve^T = \Gamma R_2 = R_{12}$

$$\begin{cases} \hat{x}(t+1) = \Phi \hat{x}(t) + K(t)\tilde{y}(t) \\ x(t+1) = \Phi x(t) + v(t) \end{cases}$$

$$\tilde{x}(t+1) = \Phi\tilde{x}(t) + v(t) - K(t)[\Theta\tilde{x}(t) + e(t)]$$

$$\underline{P(t+1)} = E[\Phi\tilde{x} + v - K\tilde{y}][\Phi\tilde{x} + v - K(\Theta\tilde{x} + e)]^T = \Phi P\Phi^T - \Phi E\tilde{x}(\Theta\tilde{x}+e)^TK^T + R_1 -$$

$$- KE(\Theta\tilde{x}+e)\tilde{x}^T\Phi^T + K\;E(\Theta\tilde{x}+e)(\Theta\tilde{x}+e)^TK^T - Ev(\Theta\tilde{x}+e)^TK^T - K(\Theta\tilde{x}+e)v^T =$$

$$= \Phi P\Phi^T - \Phi P\Theta^TK^T + \overset{+R_1)}{\Gamma R_2\Gamma^T} - K\Theta P\Phi^T + K\Theta P\Theta^TK^T + KR_2K^T - \Gamma R_2K^T - KR_2\Gamma^T =$$

$$\underline{= [\Phi - K\Theta]P(t)[\Phi - K\Theta]^T + [K - \Gamma]R_2[K - \Gamma]^T + R_1 - \Gamma R_2\Gamma^T}$$

etc.

7.4.5. $\begin{cases} x(t+1) = x(t) + b\,e(t) \\ y(t) = x(t) + e(t) \end{cases}$

$x(t_0) \in N(0, \sigma_0)$

$e(t) \in N(0, 1)$

$$\hat{x}(t+1) = E[x(t+1) \mid Y_t] = E[x(t+1) \mid Y_{t-1}] + E[x(t+1) \mid \tilde{y}(t)] - E\,x(t+1)$$

1) $E[x(t+1) \mid Y_{t-1}] = E[x(t) + b\,e(t) \mid Y_{t-1}] = E[x(t) \mid Y_{t-1}] = \hat{x}(t)$

2) $R_{x\tilde{y}} = \operatorname{cov}[x(t+1), \tilde{y}(t)] = E[x(t) + b\,e(t) - E\,x(t)][\tilde{x}(t) + e(t)] =$

$$= E[\tilde{x} + \hat{x} + b\,e(t)][\tilde{x} + e] = P(t) + b$$

$R_{\tilde{y}\tilde{y}} = \operatorname{cov}[\tilde{y}(t), \tilde{y}(t)] = E[\tilde{x} + e][\tilde{x} + e] = P + 1$

$E[x(t+1) \mid \tilde{y}] = m_x + R_{x\tilde{y}} R_{\tilde{y}\tilde{y}}^{-1} (\tilde{y} - m_{\tilde{y}}) = E\,x(t+1) + K(t)\,\tilde{y}$

$$\underline{K(t) = \frac{P(t)+b}{P(t)+1}}$$

$$\tilde{x}(t+1) = x(t+1) - \hat{x}(t+1) = x(t) + b\,e(t) - \hat{x}(t) - K(t)\,\tilde{y}(t) =$$

$$= \tilde{x}(t) + b\,e(t) - K\,\tilde{x}(t) - K\,e(t) = (1 - K(t))\,\tilde{x}(t) + (b - K(t))\,e(t)$$

$$\underline{P(t+1)} = E\,\tilde{x}\tilde{x} = E[(1-K)\tilde{x} + (b-K)e]^2 = (1-K)^2 P + (b-K)^2 =$$

$$= \left(\frac{1-b}{P+1}\right)^2 P + \left(\frac{(b-1)P}{P+1}\right)^2 = \underline{\frac{(1-b)^2}{P+1} P = d \cdot \frac{P(t)}{P(t)+1}}$$

(Eller använd 7.4.3)

$$\underline{P(t_0) = \sigma_0^2}$$

$$P(t_0+1) = d\,\frac{\sigma_0^2}{1+\sigma_0^2}$$

$$P(t_0+2) = d^2\,\frac{\sigma_0^2}{1+\sigma_0^2+d\,\sigma_0^2}$$

$$P(t_0+n) = d^n\,\frac{\sigma_0^2}{1+\sigma_0^2+d\,\sigma_0^2+d^2\sigma_0^2+\ldots+d^{n-1}\sigma_0^2}$$

1) $|d|<1$

$$P(t_0+n) = \frac{\sigma_0^2 d^n}{1+\sigma_0^2+\sigma_0^2 d+\ldots+\sigma_0^2 d^{n-1}} = \frac{\sigma_0^2 d^n}{1+\sigma_0^2\frac{1-d^n}{1-d}} = \frac{\sigma_0^2 d^n(1-d)}{1-d+\sigma_0^2(1-d^n)}$$

$$\lim_{t_0\to-\infty} P(t_0+n) = \lim_{n\to\infty} P(t_0+n) = 0$$

$$\lim_{t_0\to-\infty} K(t) = \underline{b}$$

$$\underline{\hat{x}(t+1) = \hat{x}(t) + b(y(t)-\hat{x}(t))}$$

2) $|d|>1$

$$P(t_0+n) = \frac{\sigma_0^2}{\frac{1}{d^n}+\sigma_0^2\frac{1}{d^n}+\sigma_0^2\frac{1}{d^{n-1}}+\ldots+\sigma_0^2\frac{1}{d}} = \frac{\sigma_0^2}{\frac{1}{d^n}+\sigma_0^2\sum_{i=1}^{n}\frac{1}{d^i}} = \frac{\sigma_0^2}{\frac{1}{d^n}+\sigma_0^2\frac{\frac{1}{d}(1-\frac{1}{d^n})}{1-\frac{1}{d}}}$$

$$\lim_{t_0\to-\infty} P(t_0+n) = \frac{\sigma_0^2}{\frac{\sigma_0^2\cdot\frac{1}{d}}{1-\frac{1}{d}}} = \frac{1-\frac{1}{d}}{\frac{1}{d}} = d-1 = (1-b)^2-1 = b^2-2b$$

$$\lim_{t_0\to-\infty} K(t) = \frac{b^2-2b+b}{b^2-2b+1} = \frac{b(b-1)}{(b-1)^2} = \underline{\frac{b}{b-1}}$$

$$\underline{\hat{x}(t+1) = \hat{x}(t) + \frac{b}{b-1}(y(t)-\hat{x}(t))}$$

$$\underline{\tilde{x}_1(t+1) = (1-b)\tilde{x}_1(t)}$$

$$\underline{\tilde{x}_2(t+1)} = \left(1-\frac{b}{b-1}\right)\tilde{x}_2(t) + \left(b-\frac{b}{b-1}\right)e(t) = \underline{\frac{1}{1-b}\tilde{x}_2(t) + \frac{b(b-2)}{b-1}e(t)}$$

$$\underline{\hat{y}(t+1)} = E[y(t+1)\,|\,Y_t] = E[x(t+1)+e(t+1)\,|\,Y_t] = E[x(t+1)\,|\,Y_t] = \underline{\hat{x}(t+1)}$$

$$\underline{\tilde{y}(t+1) = \tilde{x}(t+1)+e(t+1)}$$

$\underline{t\to\infty:}$ $\tilde{x}_1(t) = (1-b)\tilde{x}_1(t) \Rightarrow \tilde{x}_1(t) = 0$

$$\tilde{x}_2(t) = \frac{1}{1-b}\tilde{x}_2(t) + \frac{b(b-2)}{b-1}e(t) = -\frac{b(b-2)}{b-1}\Big/-\frac{b}{b-1}e(t) = (b-2)e(t)$$

$$\Rightarrow \tilde{y}_1(t) = e(t) \quad , \quad \tilde{y}_2(t) = (b-1)e(t)$$

7.4.6. $y(t+1) = x(t+1) + e(t+1) = x(t) + b\,e(t) + e(t+1) =$

$= y(t) - e(t) + b\,e(t) + e(t+1)$

$(1-q^{-1})y = (1+(b-1)q^{-1})e$

1) $|b-1|<1$

$A^* = 1-q^{-1}$

$C^* = 1+(b-1)q^{-1}$

$1+(b-1)q^{-1} = (1-q^{-1}) + q^{-1}g_0$

$b-1 = -1 + g_0 \Rightarrow g_0 = b$

$\hat{y}(t+1) = \frac{g_0}{1+(b-1)q^{-1}}y = \frac{b}{1+(b-1)q^{-1}}y(t)$

$\tilde{y}(t+1) = e(t+1)$

2) $|b-1|>1$

$A^* = 1-q^{-1}$

$\lambda C^* = (b-1)\left(1+\frac{1}{b-1}q^{-1}\right)$ (ekvivalent brus)

$1+\frac{1}{b-1}q^{-1} = (1-q^{-1}) + q^{-1}g_0$

$\frac{1}{b-1} = -1 + g_0 \Rightarrow g_0 = \frac{b}{b-1}$

$\hat{y}(t+1) = \frac{b/b-1}{1+\frac{1}{b-1}q^{-1}}y$

$\tilde{y}(t+1) = (b-1)e(t+1)$

Resultaten är identiska med de i föregående uppgift.

7.4.8. $\underline{\hat{x}(t+k+1|t)} = E(x(t+k+1)|Y_t) = E[\Phi x(t+k) + v(t+k) | Y_t]$

$= \Phi E[x(t+k)|Y_t] = \underline{\Phi \hat{x}(t+k|t)}$

$P(t+k+1|t) = E\tilde{x}(t+k+1)\tilde{x}^T(t+k+1)$

$\tilde{x}(t+k+1) = x(t+k+1) - \hat{x}(t+k+1|t) = \Phi x(t+k) + v(t+k) - \Phi\hat{x}(t+k|t) =$

$= \Phi\tilde{x}(t+k) + v(t+k)$

$\underline{P(t+k+1|t)} = E[\Phi\tilde{x}(t+k) + v(t+k)][\Phi\tilde{x}(t+k) + v(t+k)]^T =$

$= \underline{\Phi P(t+k)\Phi^T + R_1}$

$\underline{\hat{x}(t+k+1|t) = \Phi^k \hat{x}(t+1|t)}$

7.4.14. $\begin{cases} x(t+1) = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{\phi} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} e(t) \\ y(t) = \underbrace{(1 \quad 0)}_{\theta} x(t) + e(t) \end{cases}$

$e(t) \in N(0,1)$

$$\hat{x}(t+1) = E[x(t+1) \mid y_t] = E[x(t+1) \mid y_{t-1}, \tilde{y}] = E[x(t+1) \mid y_{t-1}] +$$
$$+ E[x(t+1) \mid \tilde{y}] - Ex(t+1).$$

1) $E[x(t+1) \mid y_{t-1}] = E[\phi x(t) + be(t) \mid y_{t-1}] = \phi \hat{x}(t)$

$$R_{x\tilde{y}} = \operatorname{cov}[x(t+1), \tilde{y}(t)] = \operatorname{cov}[\phi x(t) + be(t) \mid \theta \tilde{x} + e(t)]^T =$$

$$= E[\phi \tilde{x} + \phi \hat{x} + be][\theta \tilde{x} + e]^T = \phi P \theta^T + b$$

$$R_{\tilde{y}\tilde{y}} = E[\theta \tilde{x} + e][\theta \tilde{x} + e]^T = \theta P \theta^T + 1$$

$$E[x(t+1) \mid \tilde{y}] = m_x + R_{x\tilde{y}} R_{\tilde{y}\tilde{y}}^{-1} (\tilde{y} - m_{\tilde{y}}) = Ex(t+1) + K(t)\tilde{y}$$

$$\Rightarrow \left| \begin{array}{l} \hat{x}(t+1) = \phi \hat{x}(t) + K(t)\tilde{y} \\ \\ K(t) = [\phi P(t) \theta^T + b][\theta P(t) \theta^T + 1]^{-1} \end{array} \right.$$

$$\tilde{x}(t+1) = x(t+1) - \hat{x}(t+1) = \phi \tilde{x}(t) - K(t)[\theta \tilde{x} + e] + be$$

$$P(t+1) = E[(\phi - K\theta)\tilde{x} + (b-K)e][(\phi - K\theta)\tilde{x} + (b-K)e]^T =$$

$$= (\phi - K\theta) P(t) (\phi - K\theta)^T + (b-K)(b-K)^T$$

$$K(t) = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \left((1 \quad 0) \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right)^{-1} =$$
$$= \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) (P_1 + 1)^{-1} = \frac{1}{P_1 + 1} \begin{bmatrix} P_1 + 2P_2 \\ P_2 + 1 \end{bmatrix}$$

$$\Phi - K\Theta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - \frac{1}{p_1+1}\begin{pmatrix} p_1+2p_2 \\ p_2+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \frac{1}{p_1+1}\begin{pmatrix} p_1+1-p_1-2p_2 & 2(p_1+1) \\ -p_2-1 & p_1+1 \end{pmatrix} =$$

$$= \frac{1}{p_1+1}\begin{pmatrix} 1-2p_2 & 2(p_1+1) \\ -p_2-1 & p_1+1 \end{pmatrix}$$

$$b - K = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{p_1+1}\begin{pmatrix} p_1+2p_2 \\ p_2+1 \end{pmatrix} = \frac{1}{p_1+1}\begin{pmatrix} -p_1-2p_2 \\ p_1-p_2 \end{pmatrix}$$

Stationärt:

$$\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \frac{1}{(p_1+1)^2}\begin{pmatrix} 1-2p_2 & 2(p_1+1) \\ -(1+p_2) & p_1+1 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 1-2p_2 & -(1+p_2) \\ 2(p_1+1) & p_1+1 \end{pmatrix} + \frac{1}{(p_1+1)^2}\begin{pmatrix} -(p_1+2p_2) \\ p_1-p_2 \end{pmatrix}\begin{pmatrix} -(p_1+2p_2) & p_1-p_2 \end{pmatrix} =$$

$$= \frac{1}{(p_1+1)^2}\begin{pmatrix} p_1(1-2p_2)+2p_2(p_1+1) & p_2(1-2p_2)+2p_3(p_1+1) \\ -p_1(1+p_2)+p_2(p_1+1) & -p_2(1+p_2)+p_3(p_1+1) \end{pmatrix}\begin{pmatrix} 1-2p_2 & -(1+p_2) \\ 2(p_1+1) & (p_1+1) \end{pmatrix} +$$

$$+ \frac{1}{(p_1+1)^2}\begin{pmatrix} (p_1+2p_2)^2 & -(p_1-p_2)(p_1+2p_2) \\ -(p_1-p_2)(p_1+2p_2) & (p_1-p_2)^2 \end{pmatrix} =$$

$$= \frac{1}{(p_1+1)^2}\begin{pmatrix} p_1(1-2p_2)^2+4p_2(p_1+1)(1-2p_2)+4p_3(p_1+1)^2+(p_1+2p_2)^2 & \text{——} \\ -p_1(1+p_2)(1-2p_2)+p_2(p_1+1)(1-2p_2)-2p_2(1+p_2)(p_1+1)+2p_3(p_1+1)^2-(p_1-p_2)(p_1+2p_2) & p_1(1+p_2)^2-p_2(p_1+1)(1+p_2)-p_2(1+p_2)(p_1+1)+p_3(p_1+1)^2+(p_1-p_2)^2 \end{pmatrix}$$

Ansats: $\underline{P = \begin{pmatrix} 8 & 2 \\ 2 & 2 \end{pmatrix}}$ Stämmer!

$$\underline{K} = \frac{1}{9}\begin{pmatrix} 8+4 \\ 2+1 \end{pmatrix} = \underline{\frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix}}$$

7.5.1. $x(t+1) = \Phi x(t) + v(t)$

$y(t) = \Theta x(t) + e(t)$

$E x_0 = m$

$E(x_0 x_0^T) = R_0$

$E v v^T = R_1$

$E v e^T = 0$

$E e e^T = R_2$

$$\begin{cases} z(t) = \Phi^T z(t+1) + \Theta^T u(t+1) \\ z(t_1) = \Phi^{-1} a \end{cases}$$

elliminera $a^T x(t_1)$!

$$a^T x(t_1) = z^T(t_1)\Phi^T x(t_1) = z^T(t_0)\Phi^T x(t_0) + \sum_{t=t_0}^{t_1} \big(z^T(t)\Phi^T x(t) - $$

$$- z^T(t-1)\Phi^T x(t-1)\big)$$

$$z^T(t)\Phi^T x(t) = z^T(t)\Phi^T\Phi x(t-1) + z^T(t)\Phi^T v(t-1)$$

$$z^T(t-1)\Phi^T x(t-1) = z^T(t)\Phi\Phi^T x(t-1) + u^T(t)\Theta\Phi^T x(t-1)$$

$$a^T x(t_1) = z^T(t_0-1)\Phi^T x(t_0-1) + \sum_{t=t_0}^{t_1}\Big[z^T(t)\left[\Phi^T\Phi - \Phi\Phi^T\right]x(t-1) + z^T(t)\Phi^T v(t-1)$$

$$- u^T(t)\Theta\Phi^T x(t-1)\Big]$$

$$a^T\hat{x}(t_1) = -\sum_{t=t_0}^{t_1} u^T(t)\,y(t) \qquad = -\sum_{t=t_0}^{t_1}\big(u^T(t)\Theta x(t) + u^T(t)e(t)\big)$$

$$a^T x(t_1) - a^T\hat{x}(t_1) = z^T(t_0-1)\Phi^T x(t_0-1) \qquad + \sum_{t=t_0}^{t_1}\Big[z^T(t)\left[\Phi^T\Phi - \Phi\Phi^T\right]x(t-1) +$$

$$+ z^T(t)\Phi^T v(t-1) - u^T(t)\Theta\Phi^T x(t-1) + u^T(t)\Theta x(t) + u^T(t)e(t)\Big]$$

Slutföres icke.

7.6.1. $\begin{cases} dx = \alpha x\, dt + dv \\ dy = x\, dt + de \end{cases}$ $\qquad E(dv\, dv) = r_1\, dt$

$E(de\, de) = r_2\, dt$

$X_0 \in N(m, \sqrt{r_0})$

$$\begin{cases} d\hat{x} = \alpha \hat{x}\, dt + K(t)[dy - \hat{x}\, dt] \\ \hat{x}(t_0) = m \end{cases}$$

$$K(t) = PC^T R_2^{-1} = P r_2^{-1}$$

$$\dot{P} = \alpha P + P\alpha + r_1 - P r_2^{-1} P = 2\alpha P + r_1 - \frac{1}{r_2} P^2$$

$$\text{Ansatz: } P(t) = \frac{\frac{r_1}{\beta}\sinh\beta t + r_0[\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t]}{\cosh\beta t - \frac{\alpha}{\beta}\sinh\beta t + \frac{r_0}{r_2\beta}\sinh\beta t} \quad \leftarrow = N$$

$$\underline{\dot{P}} = \frac{1}{N^2}\Big[\Big(\cosh\beta t - \frac{\alpha}{\beta}\sinh\beta t + \frac{r_0}{r_2\beta}\sinh\beta t\Big)\Big(r_1\cosh\beta t + r_0\beta\sinh\beta t + r_0\alpha\cosh\beta t\Big) -$$

$$-\Big(\beta\sinh\beta t - \alpha\cosh\beta t + \frac{r_0}{r_2}\cosh\beta t\Big)\Big(\frac{r_1}{\beta}\sinh\beta t + r_0\cosh\beta t + \frac{\alpha r_0}{\beta}\sinh\beta t\Big)\Big] =$$

$$= \frac{1}{N^2}\Big[r_1\cosh^2\beta t + \cosh\beta t\cdot r_0[\beta\sinh\beta t + \alpha\cosh\beta t] - \frac{r_1\alpha}{\beta}\sinh\beta t\cosh\beta t -$$

$$- \frac{\alpha}{\beta} r_0 \sinh\beta t[\beta\sinh\beta t + \alpha\cosh\beta t] + \frac{r_0 r_1}{r_2\beta}\sinh\beta t\cosh\beta t +$$

$$+ \frac{r_0^2}{r_2\beta}\sinh\beta t[\beta\sinh\beta t + \alpha\cosh\beta t] - r_1\sinh^2\beta t - \beta r_0\sinh\beta t[\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t]$$

$$+ \frac{\alpha r_1}{\beta}\cosh\beta t\sinh\beta t + \alpha\cosh\beta t\, r_0[\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t] - \frac{r_0 r_1}{r_2\beta}\cosh\beta t\sinh\beta t -$$

$$- \frac{r_0^2}{r_2}\cosh\beta t[\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t] =$$

$$= \underline{\frac{1}{N^2}\Big[r_1 + \alpha r_0 + \alpha r_0 - \frac{r_0^2}{r_2}\Big]}$$

$$2\alpha P + r_1 - \frac{1}{r_2}P^2 = \frac{1}{N^2}\Big[2\alpha\Big[\frac{r_1}{\beta}\sinh\beta t + r_0(\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t)\Big]\Big[\cosh\beta t - \frac{\alpha}{\beta}\sinh\beta t +$$

$$+ \frac{r_0}{r_2\beta}\sinh\beta t\Big] + r_1\Big[\cosh\beta t - \frac{\alpha}{\beta}\sinh\beta t + \frac{r_0}{r_2\beta}\sinh\beta t\Big]^2 - \frac{1}{r_2}\Big[\frac{r_1}{\beta}\sinh\beta t + r_0[\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t]\Big]^2$$

$$= \frac{1}{N^2}\Big[2\alpha\Big(\frac{r_1}{\beta}\sin\cos - \frac{r_1\alpha}{\beta^2}\sin^2 + \frac{r_1 r_0}{r_2\beta^2}\sin^2 + r_0(\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t)\big(\frac{\alpha}{\beta}\sin + \frac{r_0}{r_2\beta}\sin\big)\Big) +$$

$$+ r_1\Big(\cos^2 + \frac{\alpha}{\beta}\sin^2 + \frac{r_0^2}{r_2^2\beta^2}\sin^2 - \frac{2\alpha}{\beta}\cos\sin + \frac{2r_0}{r_2\beta}\cos\sin - \frac{2\alpha r_0}{r_2\beta^2}\sin^2\Big) - \frac{1}{r_2}\Big[\frac{r_1^2}{\beta^2}\sinh^2\beta t +$$

$$+ \frac{2r_1 r_0}{\beta}\sinh\beta t\,(\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t) + r_0^2(\cos^2 + \frac{\alpha^2}{\beta^2}\sin^2 + \frac{2\alpha}{\beta}\cos\sin)\Big]\Big] =$$

$$= \frac{1}{N^2}\Big[r_1 + 2\alpha r_0 - \frac{r_0^2}{r_2}\Big]$$

#

7.6.2. $\begin{cases} \frac{dP}{dt} = AP + PA^T + R_1 - PC^T R_2^{-1} CP \\ P(t_0) = R_0 \end{cases}$

$$P(t) = [\Lambda_{21} + \Lambda_{22} R_0][\Lambda_{11} + \Lambda_{12} R_0]^{-1}$$

$$P[\Lambda_{11} + \Lambda_{12} R_0] = \Lambda_{21} + \Lambda_{22} R_0$$

$$\dot{P}[\Lambda_{11} + \Lambda_{12} R_0] + P[-A^T \Lambda_{11} + C^T R_2^{-1} C \Lambda_{21} - A^T \Lambda_{12} R_0 + C^T R_2^{-1} C \Lambda_{11} \overset{R_0}{]} = R_1 \Lambda_{11} + A\Lambda_{21} + $$
$$+ R_1 \Lambda_{12} R_0 + A \Lambda_{22} R_0$$

$$\dot{P}[\Lambda_{11} + \Lambda_{12} R_0] = PA^T[\Lambda_{11} + \Lambda_{12} R_0] - PC^T R_2^{-1} C[\Lambda_{21} + \Lambda_{22} R_0] + R_1[\Lambda_{11} + \Lambda_{12} R_0] + A[\Lambda_{21} + \Lambda_{22} R_0]$$

$$\underline{\dot{P} = PA^T - PC^T R_2^{-1} CP + R_1 + AP} \quad \#$$

7.6.3. $dy + ay\,dt = bu\,dt + de$

$$\begin{cases} da = -\alpha a\,dt + dv \\ db = -\beta b\,dt + dw \end{cases}$$

$$\begin{cases} x_1 = a \\ x_2 = b \end{cases} \Rightarrow \dot{x} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} x\,dt + ds \qquad E\,ds\,ds^T = \begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix} dt$$

$$dy = \begin{bmatrix} -y & u \end{bmatrix} x\,dt + de$$

$$d\hat{x} = A\hat{x}\,dt + K[dy - C\hat{x}]dt$$

$$\hat{x}(t_0) = m$$

$$K = PC^T R_2^{-1} = P \cdot \frac{1}{r}\begin{pmatrix} -y \\ u \end{pmatrix}$$

$$\dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} CP = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} P + P \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} + \begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix} - P(-y \;\; u)\frac{1}{r}\begin{pmatrix} -y \\ u \end{pmatrix} P$$

$$P(t_0) = R_0$$

(m och R_0 är okända)

7.6.5. Theorem 4.1.

$$\begin{cases} \hat{x}(t+h) = \phi \hat{x}(t) + K(t)[y(t) - \Theta \hat{x}(t)] \\ \hat{x}(t_0) = m \end{cases}$$

$$K(t) = \phi P(t) \Theta^T [\Theta P(t) \Theta^T + R_2]^{-1}$$

$$\begin{cases} P(t+h) = \phi P(t) \phi^T + R_1 h - \phi P(t) \Theta^T [\Theta P(t) \Theta^T + R_2 h]^{-1} \Theta P(t) \phi^T \\ P(t_0) = R_0 \end{cases}$$

små h:

$$\phi(t+h, t) = e^{Ah} \approx I + Ah$$

$$\Theta = \int_t^{t+h} e^{A(h-s)} C(s)\, ds \approx e^{Ah} C \cdot h \approx C \cdot h$$

$$\Rightarrow P(t+h) \approx (I + Ah) P(t) (I + Ah)^T + R_1 h - (I + Ah) P(t) C^T h [C h P(t) C^T h + R_2 h]^{-1} \cdot$$

$$\cdot C h P(t) (I + Ah)^T \approx P(t) + Ah P(t) + P(t) A^T h + R_1 h - P(t) C^T R_2^{-1} C P h$$

$$\lim_{h \to 0} \frac{P(t+h) - P(t)}{h} = A P(t) + P(t) A^T + R_1 - P(t) C^T R_2^{-1} C P(t) = \frac{dP}{dt}$$

$$\hat{x}(t+h) = (I + Ah) \hat{x}(t) + (I + Ah) P C^T h \cdot R_2^{-1} h^{-1} [y(t) - C \hat{x} h]$$

$$\hat{x}(t+h) - \hat{x}(t) = A \hat{x}(t) \cdot h + P C^T R_2^{-1} [\tfrac{dy}{dt} h - C \hat{x} h]$$

$$\dot{\hat{x}}(t) = A \hat{x}(t) + P C^T R_2^{-1} [\dot{y}(t) - C \hat{x}(t)]$$

7.6.6.

$$\begin{cases} dx = Ax\,dt + dv' + v_1\,dt \\ dy = Cx\,dt + de' + e_1\,dt \end{cases}$$

$$z = \begin{bmatrix} x \\ v_1 \\ e_1 \end{bmatrix}$$

$$\begin{cases} \underline{dz = \begin{bmatrix} A & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z\,dt + \begin{bmatrix} dv' \\ 0 \\ 0 \end{bmatrix}} \qquad z(0) = \begin{bmatrix} x(0) \\ v_1 \\ e_1 \end{bmatrix} \\ \underline{dy = \begin{bmatrix} C & 0 & I \end{bmatrix} z\,dt + de'} \end{cases}$$

7.6.7. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad R_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 \end{bmatrix} \quad R_2 = r_2$

$$\begin{cases} d\hat{x} = A\hat{x}\,dt + K[dy - C\hat{x}\,dt] \\ \hat{x}(t_0) = m \end{cases}$$

$$K = PC^T R_2^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{r_2} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \frac{1}{r_2}$$

$$\dot{P} = 0 = AP + PA^T + R_1 - PC^T R_2^{-1} CP = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{r_2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} =$$

$$= \begin{bmatrix} p_2 & p_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_2 & 0 \\ p_3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 \end{bmatrix} - \frac{1}{r_2} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 2p_2 + 1 - \frac{p_1^2}{r_2} & p_3 - \frac{p_1 p_2}{r_2} \\ p_3 - \frac{p_1 p_2}{r_2} & \sigma^2 - \frac{1}{r_2} p_2^2 \end{bmatrix}$$

$$\begin{cases} p_1^2 - 2p_2 r_2 - r_2 = 0 & \Rightarrow \underline{p_1 = \sqrt{r_2}\sqrt{2\sigma\sqrt{r_2} + 1}} \\ p_1 p_2 - r_2 p_3 = 0 & \Rightarrow \underline{p_3 = \sigma\sqrt{2\sigma\sqrt{r_2} + 1}} \\ p_2^2 - r_2 \sigma^2 = 0 & \Rightarrow \underline{p_2 = \sqrt{r_2}\,\sigma} \end{cases}$$

$$\underline{K = \frac{1}{\sqrt{r_2}} \begin{bmatrix} \sqrt{2\sigma\sqrt{r_2} + 1} \\ \sigma \end{bmatrix}}$$

7.6.7. forts.

$$d\hat{x} = A\hat{x}\,dt + K\,dy - KC\hat{x}\,dt$$

$$[S - A + KC]\,\hat{x} = KSy$$

$$\hat{x} = [S - A + KC]^{-1} KSy = \begin{pmatrix} s + k_1 & -1 \\ k_2 & s \end{pmatrix}^{-1} KSy = \frac{1}{s^2 + k_1 s + k_2}\begin{pmatrix} s & 1 \\ -k_2 & s + k_1 \end{pmatrix} KSy$$

$$\begin{cases} \hat{x}_1 = \dfrac{(sk_1 + k_2)s}{s^2 + k_1 s + k_2}\, y \\[2ex] \hat{x}_2 = \dfrac{(-k_2 k_1 + sk_2 + k_1 k_2)s}{s^2 + k_1 s + k_2}\, y = \dfrac{s^2 k_2}{s^2 + k_1 s + k_2}\, y \end{cases}$$

$$\lim_{r_2 \to 0} \frac{\hat{x}_1}{y} = \lim_{r_2 \to 0} \frac{\frac{1}{\sqrt{r_2}}\left(s\sqrt{2\sigma\sqrt{r_2} + 1} + \sigma\right)s}{\frac{1}{\sqrt{r_2}}\left(\sqrt{r_2}\,s^2 + \sqrt{2\sigma\sqrt{r_2} + 1}\; s + \sigma\right)} = \frac{(s + \sigma)s}{s + \sigma} = \underline{\underline{s}}$$

$$\lim_{r_2 \to 0} \frac{\hat{x}_2}{y} = \lim_{r_2 \to 0} \frac{\frac{1}{\sqrt{r_2}}(\sigma s^2)}{\frac{1}{\sqrt{r_2}}\left(\sqrt{r_2}\,s^2 + \sqrt{2\sigma\sqrt{r_2} + 1}\; s + \sigma\right)} = \underline{\underline{\frac{\sigma s^2}{s + \sigma}}}$$

7.6.8. $\begin{cases} dx_2 = dv_2 & R_1 = \sigma^2 \\ dz = dx_1 = x_2\,dt + dv_1 & R_2 = 1 \end{cases}$

$$z = \frac{dy}{dt}$$

$$\dot{P} = 0 = R_1 - PC^T R_2^{-1} C P = \sigma^2 - p^2 \Rightarrow \underline{P = \sigma}$$

$$\underline{K = PC^T R_2^{-1} = \sigma}$$

$$d\hat{x}_2 = A\hat{x}_2\,dt + K\,dz - KC\hat{x}_2\,dt$$

$$d\hat{x}_2 = \sigma\,dz - \sigma\hat{x}_2\,dt$$

$$(s + \sigma)\hat{x}_2 = \sigma s z$$

$$\frac{\hat{x}_2}{y} = \frac{s\hat{x}_2}{z} = \frac{\sigma s^2}{s + \sigma}$$

$$\frac{\hat{x}_1}{y} = s$$

7.6.9. $\begin{cases} dx = Ax\,dt + B\,de \\ dy = Cx\,dt + de \end{cases}$

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$$\frac{dP}{dt} = (A - R_{12}R_2^{-1}C)P + P(A - R_{12}R_2^{-1}C)^T + R_1 - R_{12}R_2^{-1}R_{12}^T - PC^TR_2^{-1}CP =$$

$$= (A - BC)P + P(A - BC)^T + BR_2B^T - BR_2B^T - PC^TR_2^{-1}CP$$