4.2.1. $y(t)+ay(t-1)=e(t)+ce(t-1)$

$e(t)\in N(0,1)$

$$H(q)=\frac{1+cq^{-1}}{1+aq^{-1}}=1+\frac{(c-a)q^{-1}}{1+aq^{-1}}=1+(c-a)\sum_{k=0}^{\infty}(-1)^k a^k q^{(-k-1)}$$

$$H(q)=\sum_{k=0}^{\infty}h(k)q^{-k}$$

$$\Rightarrow\begin{cases}h(0)=1\\ h(k)=(c-a)(-1)^{k-1}a^{k-1} & k\geq 1\end{cases}$$

$$r_y(\tau)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}h(k)h(l)r_u(\tau+l-k)=$$

$$=\sum_{k=0}^{\infty}h(k)h(k-\tau)=h(\tau)h(0)+\sum_{k=\tau+1}^{\infty}(c-a)^2(-a)^{2k-\tau-2}=$$

$$=h(\tau)+(c-a)^2\cdot(-a)^{-\tau-2}\sum_{k=\tau+1}^{\infty}(-a)^{2k}=[\tau\neq 0]$$

$$=(c-a)(-a)^{\tau-1}+(c-a)^2(-a)^{-\tau-2}\frac{(-a)^{2\tau+2}}{1-a^2}$$

$$=(c-a)(-a)^{\tau-1}+(-a)^{\tau}\frac{(c-a)^2}{1-a^2}=\underline{\left(1-\frac{c}{a}\right)\frac{1-ac}{1-a^2}(-a)^{\tau}}\quad \tau\neq 0$$

$$r_y(0)=h(0)+(c-a)^2(-a)^{-2}\sum_{k=1}^{\infty}(-a)^{2k}=$$

$$=1+\frac{(c-a)^2}{(-a)^2}\cdot\frac{a^2}{1-a^2}=\frac{1-a^2+c^2-2ca+a^2}{1-a^2}=$$

$$=\underline{\frac{1-2ac+c^2}{1-a^2}}$$

$$r_{ey}(\tau)=\sum_{k=0}^{\infty}h(k)r_e(\tau+k)=\begin{cases}\underline{0\quad \tau>0}\\ \underline{1\quad \tau=0}\\ \underline{(c-a)(-a)^{-\tau-1}\quad \tau<0}\end{cases}$$

4.2.4. $y(t) + a(t-1)y(t-1) = u(t)$

$$m_y(t) = -a(t-1)\,Ey(t-1) + Eu(t) = -a(t-1)\,m_y(t-1) + m_u(t)$$

$$r_{uy}(s,t) = E[u(s) - m_u(s)][y(t) - m_y(t)] =$$

$$= E[u(s) - m_u(s)][-a(t-1)y(t-1) + u(t) + a(t-1)m_y(t-1) -$$

$$- m_u(t)] = E[u(s) - m_u(s)][u(t) - m_u(t)] +$$

$$+ a(t-1)\,E[u(s) - m_u(s)][m_y(t-1) - y(t-1)] =$$

$$= \underline{r_u(s,t) - a(t-1)\,r_{uy}(s,t-1)}$$

$$r_{yu}(s,t) = E[y(s) - m_y(s)][u(t) - m_u(t)] = E[-a(s-1)y(s-1) + u(s) +$$

$$+ a(s-1)m_y(s-1) - m_u(s)][u(t) - m_u(t)] =$$

$$= \underline{-a(s-1)\,r_{yu}(s-1,t) + r_u(s,t)}$$

$$r_y(s,t) = E[y(s) - m_y(s)][y(t) - m_y(t)] = E[-a(s-1)y(s-1) + u(s)$$

$$+ a(s-1)m_y(s-1) - m_u(s)][-a(t-1)y(t-1) + u(t) +$$

$$+ a(t-1)m_y(t-1) - m_u(t)] = a^2(s-1)(t-1)\,E\big(y(s-1) - m_y(s-1)\big)\cdot$$

$$\cdot\big(y(t-1) - m_y(t-1)\big) + r_u(s,t) =$$

$$= \underline{a^2(s-1)(t-1)\,r_y(s-1,t-1) + r_u(s,t)} \quad \text{etc.}$$

4.3.1. $\Phi(\omega) = \frac{2+2\cos\omega}{5+4\cos\omega} = \frac{(1+e^{i\omega})(1+e^{-i\omega})}{(2+e^{i\omega})(2+e^{-i\omega})}$

$$\Phi(z) = \frac{(1+z)(z+1)}{(2+z)(2z+1)}$$

$$H(z) = \frac{1+z}{2z+1}$$

4.3.2. $y(t) = e(t) + 4e(t-1)$

$$x(t) = \lambda(\varepsilon(t) + c\,\varepsilon(t-1))$$

$$\Phi_y(q^{-1}) = \frac{1+4q^{-1}}{1} \cdot \frac{1+4q}{1}$$

$$\Phi_x(q^{-1}) = \lambda^2 \frac{1+cq^{-1}}{1} \cdot \frac{1+cq}{1}$$

$$(1+4q^{-1})(1+4q) = \lambda^2(1+cq^{-1})(1+cq)$$

$$1+4q^{-1}+4q+16 = \lambda^2(1+cq^{-1}+cq+c^2)$$

$$\begin{cases} 17 = \lambda^2(1+c^2) \\ 4 = \lambda^2 \cdot c \end{cases}$$

$$17 = \frac{4}{c}(1+c^2) = \frac{4}{c} + 4c$$

$$4c^2 + 4 - 17c = 0$$

$$c = \frac{17}{8} \pm \sqrt{\frac{289}{64} - \frac{64}{64}} = \frac{17}{8} \pm \sqrt{\frac{225}{64}} = \frac{17}{8} \pm \frac{15}{8}$$

$$\begin{cases} c = \frac{1}{4} \Rightarrow \lambda = \pm 4 \\ c = 4 \Rightarrow \lambda = \pm 1 \end{cases}$$

4.3.3. $y(t) = x_1(t) + x_2(t)$

$$\begin{cases} x_1(t+1) = -a x_1(t) + v_1(t) & v_1 \in N(0, \sigma_1) \\ x_2(t+1) = -b x_2(t) + v_2(t) & v_2 \in N(0, \sigma_2) \end{cases}$$

$m_y(t) = 0$

$\Phi_1(\omega) = \sigma_1^2 \qquad \Phi_2(\omega) = \sigma_2^2$

$$\Phi_y(q) = \frac{1}{(a+q)} \cdot \frac{1}{(a+q^{-1})} \sigma_1^2 + \frac{1}{(b+q)} \cdot \frac{1}{(b+q^{-1})} \sigma_2^2 =$$

$$= \frac{(b+q)(b+q^{-1})\sigma_1^2 + (a+q)(a+q^{-1})\sigma_2^2}{(a+q)(a+q^{-1})(b+q)(b+q^{-1})}$$

$$\lambda^2 (q+c)(q^{-1}+c) = \lambda^2(1 + c^2 + qc + q^{-1}c) = \sigma_1^2(b^2 + 1 + bq + bq^{-1}) +$$

$$+ \sigma_2^2(a^2 + 1 + aq + aq^{-1}) = \sigma_1^2(b^2+1) + \sigma_2^2(a^2+1) + q(\sigma_1^2 b + \sigma_2^2 a) +$$

$$+ q^{-1}(\sigma_1^2 b + \sigma_2^2 a)$$

$$\Rightarrow \begin{cases} \lambda^2 c = \sigma_1^2 b + \sigma_2^2 a \\ \lambda^2(1+c^2) = \sigma_1^2(b^2+1) + \sigma_2^2(a^2+1) \end{cases}$$

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4.3.4. $\begin{cases} x(t+1) = 0.8x(t) - 1.2e(t) \\ y(t) = x(t) + e(t) \end{cases}$

$$y(t+1) = 0.8x(t) - 1.2e(t) + e(t+1) = 0.8y(t) - 0.8e(t) - 1.2e(t) +$$

$$+ e(t+1) = 0.8y(t) - 2e(t) + e(t+1).$$

$$\Phi_y(q) = \frac{(-2+q)}{(-0.8+q)} \cdot \frac{(-2+q^{-1})}{(-0.8+q^{-1})}$$

$\begin{cases} x(t+1) = 0.8x(t) + 0.6e(t) \\ z(t) = x(t) + 2e(t) \end{cases}$

$$z(t+1) = 0.8x(t) + 0.6e(t) + 2e(t+1) = 0.8z(t) - 1.6e(t) +$$

$$+ 0.6e(t) + 2e(t+1) = 0.8z(t) - e(t) + 2e(t+1)$$

$$\Phi_z(q) = \frac{(-1+2q)}{(-0.8+q)} \cdot \frac{(-1+2q^{-1})}{(-0.8+q^{-1})} = \frac{(-q^{-1}+2)(-q+2)}{(-0.8+q)(-0.8+q^{-1})} =$$

$$= \frac{(-2+q)(-2+q^{-1})}{(-0.8+q)(-0.8+q^{-1})} = \Phi_y(q)$$

4.5.1. $$\Phi(\omega) = \frac{\omega^2+1}{\omega^4+8\omega^2+4} = \frac{\omega^2+1}{(\omega^2+4-\sqrt{12})(\omega^2+4+\sqrt{12})} =$$

$$= \frac{(1+i\omega)(1-i\omega)}{(\sqrt{4-\sqrt{12}}+i\omega)(\sqrt{4-\sqrt{12}}-i\omega)(\sqrt{4+\sqrt{12}}+i\omega)(\sqrt{4+\sqrt{12}}-i\omega)}$$

$$G(s) = \frac{1+s}{(\sqrt{4-\sqrt{12}}+s)(\sqrt{4+\sqrt{12}}+s)} = \frac{1+s}{(\alpha+s)(\beta+s)}$$

$$h(t) = \mathcal{L}^{-1}G(s) = \mathcal{L}^{-1}\frac{s+1}{(s+\alpha)(s+\beta)} = \frac{(\alpha-1)e^{-\alpha t}-(\beta-1)e^{-\beta t}}{\alpha-\beta}$$

$$Y(t) = \int_{-\infty}^{t} \frac{(\alpha-1)e^{-\alpha(t-s)}-(\beta-1)e^{-\beta(t-s)}}{\alpha-\beta}\,dw(s)$$

4.5.2. $$r(\tau) = e^{-|\tau|}\cos 2\tau$$

$$e^{-|\tau|} \to \frac{1}{\pi}\frac{1}{1+\omega^2}$$

$$\cos 2\tau \to \frac{1}{2}\left(\delta(\omega+2)+\delta(\omega-2)\right)$$

$$e^{-|\tau|}\cos 2\tau \to \Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+(\omega-\omega')^2}\left(\delta(\omega'+2)+\delta(\omega'-2)\right)d\omega' =$$

$$= \frac{1}{2\pi}\left(\frac{1}{1+(\omega-2)^2}+\frac{1}{1+(\omega+2)^2}\right) =$$

$$= \frac{1}{2\pi}\left(\frac{1}{\omega^2-4\omega+5}+\frac{1}{\omega^2+4\omega+5}\right) = \frac{1}{2\pi}\frac{2(\omega^2+5)}{(\omega^2-4\omega+5)(\omega^2+4\omega+5)} =$$

$$= \frac{1}{2\pi}\frac{2(\sqrt{5}+i\omega)(\sqrt{5}-i\omega)}{(i\omega+2i+1)(i\omega+2i-1)(-i\omega+2i+1)(-i\omega+2i-1)}$$

$$G(s) = \frac{\sqrt{1/\pi}\,(s+\sqrt{5})}{(s+2i+1)(s+2i-1)} = \frac{\sqrt{1/\pi}\,(s+\sqrt{5})}{s^2+4is-5}$$

4.5.3. $\phi_x = \frac{1}{1+\omega^2}$ $\qquad \phi_y = \frac{1}{4+\omega^2}$ $\qquad \phi_{xy} = \frac{1}{\omega^2+i\omega+2}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \int_{-\infty}^{t} \begin{pmatrix} h_x(t-s) \\ h_y(t-s) \end{pmatrix} dv(s)$$

$$\phi_{yx} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-in\omega} r_{yx}(n) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-in\omega} r_{xy}(-n) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{in\omega} r_{xy}(n) =$$

$$= \overline{\phi_{xy}}$$

$$\Phi = \begin{bmatrix} \phi_x & \phi_{xy} \\ \phi_{yx} & \phi_y \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+i\omega)(1-i\omega)} & \frac{1}{(2-i\omega)(1+i\omega)} \\ \frac{1}{(2+i\omega)(1-i\omega)} & \frac{1}{(2+i\omega)(2-i\omega)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{1+i\omega} \\ \frac{1}{2+i\omega} \end{bmatrix} \begin{bmatrix} \frac{1}{1-i\omega} & \frac{1}{2-i\omega} \end{bmatrix} = G(i\omega)\, G^T(-i\omega)$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \int_{-\infty}^{t} \begin{bmatrix} e^{-(t-s)} \\ e^{-2(t-s)} \end{bmatrix} dv(s)$$