3.3.1.

$$x(t+1) = \begin{bmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{bmatrix} x(t) \qquad h = \frac{\pi}{4n}$$

$$m_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad R_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\phi^t = \begin{bmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{bmatrix}^t = \exp\begin{bmatrix} 0 & ht \\ -ht & 0 \end{bmatrix} = \begin{bmatrix} \cos ht & \sin ht \\ -\sin ht & \cos ht \end{bmatrix}$$

$$P(t) = \phi^t R_0 (\phi^T)^t = \begin{bmatrix} \cos ht & \sin ht \\ -\sin ht & \cos ht \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos ht & -\sin ht \\ \sin ht & \cos ht \end{bmatrix} =$$

$$= \begin{bmatrix} \cos ht - \sin ht & -\cos ht + \sin ht \\ -\sin ht - \cos ht & \sin ht + \cos ht \end{bmatrix}\begin{bmatrix} \cos ht & -\sin ht \\ \sin ht & \cos ht \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2 - \cos\sin - \cos\sin + \sin^2 & -\cos\sin + \sin^2 - \cos^2 + \cos\sin \\ -\cos\sin - \cos^2 + \sin^2 + \cos\sin & \sin^2 + \cos\sin + \cos\sin + \cos^2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 - 2\cos ht \sin ht & \sin^2 ht - \cos^2 ht \\ \sin^2 ht - \cos^2 ht & 1 + 2\cos ht \sin ht \end{bmatrix}$$

$x_1(t^*)$ och $x_2(t^*)$ är oberoende om $\sin^2 ht^* - \cos^2 ht^* = 0$

$$ht^* = \frac{\pi}{4n} t^* = \frac{\pi}{4} \Rightarrow \underline{\underline{t^* = n}}$$

$$\underline{m(t^*)} = \begin{bmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}}}$$

$$\underline{P(t^*)} = \begin{bmatrix} 1 - 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 + 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}}$$

3.3.2. $x(t+1) = \begin{bmatrix} 1.5 & 1 \\ -0.7 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} e(t)$

$e(t) \in N(0,1)$

$$R_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \begin{pmatrix} 1 & 0.5 \end{pmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix}$$

$$P_\infty = \Phi P_\infty \Phi^T + R_1$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ -0.7 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 1.5 & -0.7 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix} =$$

$$= \begin{bmatrix} 1.5P_{11} + P_{21} & 1.5P_{12} + P_{22} \\ -0.7P_{11} & -0.7P_{12} \end{bmatrix} \begin{bmatrix} 1.5 & -0.7 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix} =$$

$$= \begin{bmatrix} 2.25P_{11} + 1.5P_{21} + 1.5P_{12} + P_{22} & -1.05P_{11} - 0.7P_{21} \\ -1.05P_{11} - 0.7P_{12} & 0.49P_{11} \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix}$$

$P_{12} = P_{21}$

$$\Rightarrow P_\infty \approx \begin{bmatrix} 18.85 & -11.37 \\ -11.37 & 9.49 \end{bmatrix}$$

3.3.3. $x(t+1) = a x(t) + e(t) \qquad |a|<1$

$e(t) \in N(0, \sigma)$

$x(t_0) \in N(0, \sigma_0)$

$P(t_0) = \sigma_0^2$

$P(t+1) = a^2 P(t) + R_1 = a^2 P(t) + \sigma^2$

$$P(t) = a^{2(t-t_0)}\sigma_0^2 + \sum_{k=t_0}^{t-1} a^{2(k-t_0)}\sigma^2 = a^{2(t-t_0)}\sigma_0^2 + \sigma^2 \cdot \frac{1-a^{2(t-t_0)}}{1-a^2}$$

$$\lim_{t\to\infty} P(t) = \frac{\sigma^2}{1-a^2}$$

$$\sigma_0^2 = \frac{\sigma^2}{1-a^2}$$

$$P(t) = \frac{\sigma^2}{1-a^2} a^{2(t-t_0)} + \frac{\sigma^2}{1-a^2}\left(1 - a^{2(t-t_0)}\right) = \frac{\sigma^2}{1-a^2}$$

$$R(s,t) = \Phi(s,t) P(t) = \frac{\sigma^2}{1-a^2} \cdot a^{|s-t|}$$ $\Rightarrow$ svagt stationär process

$$\Phi(\omega) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{\sigma^2}{1-a^2} a^{|n|} e^{-in\omega} = \frac{\sigma^2}{2\pi(1-a^2)}\left(\sum_0^\infty a^n e^{in\omega} + \sum_0^\infty a^n e^{-in\omega} - \right.$$

$$= \frac{\sigma^2}{2\pi(1-a^2)}\left(\frac{1}{1-ae^{i\omega}} + \frac{1}{1-ae^{-i\omega}} - 1\right) =$$

$$= \frac{\sigma^2}{2\pi(1-a^2)}\left(\frac{1-ae^{-i\omega}+1-ae^{i\omega}-1-a^2+ae^{-i\omega}+ae^{i\omega}}{1+a^2-ae^{-i\omega}-ae^{i\omega}}\right) =$$

$$= \frac{\sigma^2}{2\pi(1-a^2)} \cdot \frac{1-a^2}{1+a^2-2a\cos\omega} = \frac{\sigma^2}{2\pi(1+a^2-2a\cos\omega)}$$

3.5.1. $$I = \int_0^t f(s)\,dy(s)$$

$$E\,y(t) = m(t) \qquad \operatorname{cov}[dy, dy] = dt$$

$$y(0) = 0$$

$$I = f(t)y(t) - f(0)y(0) - \int_0^t f'(s)\,y(s)\,ds = f(t)y(t) - \int_0^t f'(s)\,y(s)\,ds$$

$$EI = f(t)\,m(t) - \int_0^t f'(s)\,m(s)\,ds = f(t)\,m(t) - f(t)\,m(t) + f(0)\,m(0)$$

$$+ \int_0^t f(s)\,m'(s)\,ds = \int_0^t f(s)\,m'(s)\,ds$$

$$\operatorname{Var} I = \operatorname{Var}\left[f(t)\,y(t) - \int_0^t f'(s)\,y(s)\,ds\right] = \operatorname{Var}\left[f(t)\,y(t)\right] +$$

$$+ \operatorname{Var}\left[\int_0^t f'(s)\,y(s)\,ds\right] - 2\operatorname{cov}\left[f(t)\,y(t), \int_0^t f'(s)\,y(s)\,ds\right] = \ldots$$

3.6.3. $dx = \alpha x\,dt + dw$

$$\frac{dm}{dt} = \alpha m \Rightarrow \underline{m(t) = m_0 e^{+\alpha(t-t_0)}}$$

$$\begin{cases} \frac{dP}{dt} = 2\alpha P + r_1 \\ P(t_0) = r_0 \end{cases}$$

Homogen lsg: $P(t) = A e^{2\alpha t}$

Partikulär lsg: $P(t) = -\frac{r_1}{2\alpha}$ $(\alpha \neq 0)$

$$P(t) = A e^{2\alpha t} - \frac{r_1}{2\alpha} = \left(r_0 + \frac{r_1}{2\alpha}\right) e^{2\alpha(t-t_0)} - \frac{r_1}{2\alpha}$$

$$\underline{R(s,t)} = e^{\alpha(s-t)} P(t) = \underline{\left(r_0 + \frac{r_1}{2\alpha}\right) e^{\alpha(t-2t_0+s)} - \frac{r_1}{2\alpha} e^{\alpha(s-t)}}$$

$\lim_{t\to\infty} P(t) = -\frac{r_1}{2\alpha}$ och existerar endast om $\underline{\underline{\alpha < 0}}$

$$r_0 = -\frac{r_1}{2\alpha}$$

$$R(s,t) = -\frac{r_1}{2\alpha} e^{\alpha(s-t)}$$

m konstant, R beror endast av $s-t$ $\Rightarrow$ stationär

$$\Phi(\omega) = \frac{1}{2\pi} \cdot \frac{-r_1}{2\alpha} \cdot \frac{-2\alpha}{\omega^2+\alpha^2} = \underline{\underline{\frac{r_1}{2\pi} \cdot \frac{1}{\omega^2+\alpha^2}}}$$

3.6.2. $$dx = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} x\,dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dv$$

$$a_1 > 0 \qquad a_2 > 0$$

$$\frac{dP}{dt} = AP + PA^T + R_1 = 0$$

$$\begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} -a_1 & 1 \\ -a_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$\begin{cases} -a_1 p_1 - a_2 p_2 - p_1 a_1 - p_2 a_2 + 1 = 0 \\ p_1 - a_1 p_2 - a_2 p_3 = 0 \\ -a_1 p_2 - a_2 p_3 + p_1 = 0 \\ p_2 + p_2 = 0 \end{cases} \quad \Rightarrow p_2 = 0$$

$$-2a_1 p_1 + 1 = 0 \Rightarrow p_1 = \frac{1}{2a_1}$$

$$p_3 = \frac{1}{2a_1 a_2}$$

$$\underline{P_\infty = \begin{pmatrix} \frac{1}{2a_1} & 0 \\ 0 & \frac{1}{2a_1 a_2} \end{pmatrix}}$$

3.6.5. $dx = Ax\,dt + b\,dv$

$$\frac{dP}{dt} = AP + PA^T + R_1 = AP + PA^T + bb^T = 0$$

$$z(t) = e^{At}b \qquad z^T(t) = b^T e^{A^T t}$$

$$R = \int_0^\infty z(t)z^T(t)\,dt$$

$$AR + RA^T + bb^T = \int_0^\infty A e^{At} bb^T e^{A^T t}\,dt + \int_0^\infty e^{At} bb^T e^{A^T t} A^T\,dt + bb^T =$$

$$= \int_0^\infty \left(A e^{At} bb^T e^{A^T t} + e^{At} bb^T e^{A^T t} A^T\right)dt + bb^T =$$

$$= \left[e^{At} bb^T e^{A^T t} \right]_0^\infty + bb^T = -bb^T + bb^T = \underline{0}$$

#

3.6.6. $\begin{cases} dx = Ax\,dt + b\,dv \\ y = x_1 \end{cases}$

$$z(t) = e^{At}b$$

$$P_\infty = \int_0^\infty z(t)\,z^T(t)\,dt \quad \text{enl. 3.6.5.}$$

$$R(s,t) = e^{A(s-t)} P_\infty$$

$$r_y(\tau) = (1\,0\,0\,0\ldots0)\, e^{A\tau} \cdot P_\infty \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

?

3.6.7. ① $dx = Axdt + dv$

Eul. 3.6.5: $P_\infty = \int_0^\infty e^{At} R_1 e^{A^T t} dt$

② $\left.\begin{aligned} \frac{dx}{dt} &= A^T x \\ x(0) &= b \end{aligned}\right\} \Rightarrow x(t) = e^{A^T t} b$

$$V = \int_0^\infty x^T(t) R_1 x(t)\, dt = \int_0^\infty b^T e^{At} R_1 e^{A^T t} b = [R_1 = I] = b^T \int_0^\infty e^{At} R_1 e^{A^T t} dt\, b$$

$$\underline{\underline{V = b^T P_\infty b}}$$

3.6.8. $\begin{cases} m \frac{dv}{dt} + fV = K(t) \\ E(V^2) = \frac{kT}{m} \end{cases}$

$dv = -\frac{f}{m} v\, dt + \frac{1}{m} K(t)$

$\underline{E\, v(t) = 0}$

$\begin{cases} \frac{dP}{dt} = -\frac{2f}{m} P + r_1 \\ P(0) = 0 \end{cases} \Rightarrow P(t) = \frac{m}{2f} r_1 \left(1 - e^{-\frac{2f}{m} t}\right)$

$P(\infty) = \frac{kT}{m} = \frac{m r_1}{2f} \Rightarrow \underline{P(t) = \frac{kT}{m}\left(1 - e^{-\frac{2f}{m} t}\right)}$

3.6.9. $dx = A(t)\,x\,dt + dv$

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,s)\,dv(s) =$$

$$= \Phi(t,t_0)\,x(t_0) + v(t) - \Phi(t,t_0)\,v(t_0) - \int_{t_0}^{t} \left[\frac{d}{ds}\Phi(t,s)\right] v(s)\,ds$$

$$m_x(t) = \Phi(t,t_0)\,m_x(t_0) = \Phi(t,t_0)\,m_0$$

$$\underline{\frac{dm_x(t)}{dt}} = \frac{d\Phi(t,t_0)}{dt}\cdot m_0 = A(t)\,\Phi(t,t_0)\,m_0 = \underline{A(t)\,m_x(t)}$$

$m_0 = 0$

$$\underline{R(s,t)} = \operatorname{cov}[x(s), x(t)] = [s \geq t] = \operatorname{cov}\Big[\Big(\Phi(s,t)\,x(t) + v(s) - \Phi(s,t)\,v(t) -$$

$$- \int_{s}^{t} \left[\frac{d}{d\tau}\Phi(t,\tau)\right] v(\tau)\,d\tau\Big)\,x^T(t)\Big] =$$

$$= \Phi(s,t)\,E\,x(t)\,x^T(t) = \underline{\Phi(s,t)\,P(t)}$$

3.7.1. $\begin{cases} \dot{x}_1 = e & x_1(0) = 0 \\ \dot{x}_2 = x_1 e & x_2(0) = 0 \end{cases}$

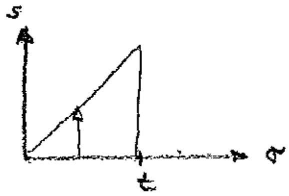
$E e(t) = 0 \qquad \operatorname{cov}[e(s), e(t)] = r(\tau)$

$$x_1(t) = \int_0^t e(t)\,dt$$

$$m_2'(t) = E\dot{x}_2(t) = E\int_0^t e(s)\,ds\, e(t) = \int_0^t E e(s) e(t)\,ds = \int_0^t r(t-s)\,ds$$

$$\underline{m_2(t)} = \int_0^t m_2'(\sigma)\,d\sigma = \int_0^t \int_0^\sigma r(\sigma - s)\,ds\,d\sigma = [r(\tau) = r(-\tau)] =$$

$$= \int_0^t \int_0^\sigma r(s)\,ds\,d\sigma = \int_0^t \int_s^t r(s)\,d\sigma\,ds =$$



$$= \underline{\int_0^t (t-s) r(s)\,ds}$$

$$\underline{r(s) \to \delta(s) \implies m_2(t) \to t}$$

3.7.2. $x_2(t) = \int_0^t w(s)\,dw(s)$

$$I_\lambda = (1-\lambda) I_0 + \lambda I_1 = (1-\lambda) \lim \sum_{i=1}^N w(t_i)\left[w(t_{i+1}) - w(t_i)\right] +$$

$$+ \lambda \lim \sum_{i=1}^N w(t_{i+1})\left[w(t_{i+1}) - w(t_i)\right] =$$

$$= \lim \sum_{i=1}^N \left[(1-\lambda) w(t_i) + \lambda w(t_{i+1})\right]\left[w(t_{i+1}) - w(t_i)\right] =$$

$$= \lim \sum_{i=1}^N \Big[w(t_i) w(t_{i+1}) - \lambda w(t_i) w(t_{i+1}) - w^2(t_i) + \lambda w^2(t_i) + \lambda w^2(t_{i+1}) -$$

$$- \lambda w(t_{i+1}) w(t_i)\Big] =$$

$$= \lim \sum_{i=1}^{N} \left[\frac{1}{2}\left(W^2(t_{i+1}) - W^2(t_i)\right) + \left(\lambda - \frac{1}{2}\right)\left[W(t_{i+1}) - W(t_i)\right]^2 \right] =$$

$$= \frac{1}{2}\left(W^2(t) - W^2(0)\right) + \left(\lambda - \frac{1}{2}\right) t$$

$$E(X_2(t)) = \frac{1}{2}t + \lambda t - \frac{1}{2}t = \underline{\underline{\lambda t}}$$

3.7.3. $\begin{cases} \Delta X_1(t) = \Delta W(t) \\ \Delta X_2(t) = X_1(t)\,\Delta W(t) \end{cases}$ $\qquad \underline{\underline{E(X_1(t)) = E(W(t)) = 0}}$

a) $\Delta f(t) = f(t+h) - f(t)$

$$X_1(t+h) - X_1(t) = W(t+h) - W(t)$$

$$X_2(t+h) = X_2(t) + W(t)\left(W(t+h) - W(t)\right)$$

$$E(X_2(t+h)) = E(X_2(t)) \Rightarrow \underline{\underline{E X_2(t) = 0}}$$

b) $\Delta f(t) = f(t) - f(t-h)$

$$X_2(t+h) = X_2(t) + W(t)\left(W(t) - W(t-h)\right)$$

$$E X_2(t+h) = E X_2(t) + t - (t-h) = E X_2(t) + h$$

$$\Rightarrow \underline{\underline{E X_2(t) = t}}$$

c) $\Delta f(t) = \frac{1}{2}\left(f(t+h) - f(t-h)\right)$

$$X_2(t+h) = X_2(t) + W(t)\cdot\frac{1}{2}\left(W(t+h) - W(t-h)\right)$$

$$E X_2(t+h) = E X_2(t) + \frac{1}{2}t - \frac{1}{2}(t-h) = E X_2(t) + \frac{1}{2}h$$

$$\Rightarrow \underline{\underline{E X_2(t) = \frac{1}{2}t}}$$

3.7.4. $\begin{cases} x_1(t+h) = x_1(t) + h\,e(t) \\ x_2(t+h) = x_2(t) + h\,x_1(t)\,e(t) \end{cases}$

$$x_1(0) = 0$$

$$x_1(h) = h\,e(0)$$

$$x_1(2h) = h\,e(0) + h\,e(h)$$

$$\vdots$$

$$\underline{x_1(Nh) = h\sum_{i=0}^{N-1} e(ih)}$$

$$x_2(0) = 0$$

$$x_2(h) = 0$$

$$x_2(2h) = h \cdot h\,e(0) \cdot e(h)$$

$$x_2(3h) = h^2 e(0)e(h) + h^2 (e(0) + e(h))\,e(2h)$$

$$\vdots$$

$$\underline{x_2(Nh) = h^2 \sum_{i=1}^{N-1} e(ih) \sum_{j=0}^{i-1} e(jh)}$$

$$\underline{E\,x_1(Nh)} = h\sum_{i=0}^{N-1} E\,e(ih) = \underline{0}$$

$$\underline{E\,x_2(Nh)} = h^2 \sum_{i=1}^{N-1}\sum_{j=0}^{i-1} E\,e(ih)\,e(jh) = h^2 \sum_{i=1}^{N-1}\sum_{j=0}^{i-1} r((i-j)h)$$

$$= h^2 \sum_{i=1}^{N-1}\sum_{\mu=1}^{i} r(\mu h) = \underline{h^2 \sum_{i=1}^{N-1} (N-i)\,r(ih)}$$

3.8.1. $\begin{cases} dx = \sqrt{x}\, dW + \frac{1}{4}\, dt \\ x(0) = 1 \end{cases}$

$$x(t) = \left(1 + \tfrac{1}{2} W(t)\right)^2$$

$$f(W) = \left(1 + \tfrac{1}{2} W(t)\right)^2$$

$$\underline{dx} = \tfrac{1}{2} f_{WW}\, dt + f_W\, dW = \tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\, dt + 2\left(1 + \tfrac{1}{2} W\right) \cdot \tfrac{1}{2}\, dW =$$

$$= \underline{\tfrac{1}{4}\, dt + \sqrt{x}\, dW}$$

3.8.2. $\begin{cases} dx_1 = x_2\, dW - \frac{1}{2} x_1\, dt & x_1(0) = 0 \\ dx_2 = -x_1\, dW - \frac{1}{2} x_2\, dt & x_2(0) = 1 \end{cases}$

$$\begin{cases} x_1(t) = \sin W(t) \\ x_2(t) = \cos W(t) \end{cases}$$

$$f(W) = \begin{bmatrix} \sin W \\ \cos W \end{bmatrix}$$

$$\underline{dx} = \tfrac{1}{2} f_{WW}\, dt + f_W\, dW = \tfrac{1}{2} \begin{bmatrix} -\sin W \\ -\cos W \end{bmatrix} dt + \begin{bmatrix} \cos W \\ -\sin W \end{bmatrix} dW =$$

$$= \underline{\tfrac{1}{2} \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} dt + \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} dW}$$

3.8.3. $\begin{cases} dx_1 = x_2\, dt & x_1(0) = 1 \\ dx_2 = dW & x_2(0) = 0 \end{cases}$

$$x(t) = f(W(t))$$

$$dx = \frac{1}{2} f_{WW}\, dt + f_W\, dW = \frac{1}{2}\begin{pmatrix} f^1_{WW} \\ f^2_{WW} \end{pmatrix} dt + \begin{pmatrix} f^1_W \\ f^2_W \end{pmatrix} dW = \begin{pmatrix} f^2\, dt \\ dW \end{pmatrix}$$

$$\frac{1}{2} f^2_{WW}\, dt + f^2_W\, dW = dW \Rightarrow \underline{f^2 = W}$$

$$\frac{1}{2} f^1_{WW}\, dt + f^1_W\, dW = W\, dt$$

3.9.1. $J \dfrac{d^2\varphi}{dt^2} + D \dfrac{d\varphi}{dt} + C\varphi = M$

$$\begin{cases} \varphi_1 = \varphi \\ \varphi_2 = \dot\varphi \end{cases} \qquad \begin{cases} \dot\varphi_1 = \varphi_2 \\ \dot\varphi_2 = -\frac{D}{J}\varphi_2 - \frac{C}{J}\varphi_1 + \frac{M}{J} \end{cases}$$

$$\begin{cases} d\varphi_1 = \varphi_2\, dt \\ d\varphi_2 = \left[-\frac{C}{J}\varphi_1 - \frac{D}{J}\varphi_2\right]dt + \frac{1}{J}dW \end{cases}$$

$$\frac{dP}{dt} = AP + PA^T + R_1 = 0$$

$$\begin{bmatrix} \dot P_1 & \dot P_2 \\ \dot P_2 & \dot P_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{C}{J} & -\frac{D}{J} \end{bmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} 0 & -\frac{C}{J} \\ 1 & -\frac{D}{J} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{J}r_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} 0 = P_2 + P_2 \quad \Rightarrow \underline{P_2 = 0} \\ 0 = P_3 - \frac{C}{J}P_1 \\ 0 = -\frac{C}{J}P_1 + P_3 \\ 0 = -\frac{D}{J}P_3 - \frac{D}{J}P_3 + \frac{1}{J}r_1 \end{cases}$$

$$E\varphi^2 = \underline{P_1 = \frac{kT}{C}}$$

$$E\dot\varphi^2 = \underline{P_3 = \frac{kT}{J}}$$

$$-2\frac{D}{J}P_3 + \frac{1}{J}r_1 = -\frac{2DkT}{J^2} + \frac{1}{J}r_1 = 0 \quad \Rightarrow \underline{r_1 = \frac{2DkT}{J}}$$

$$\underline{R(\tau) = e^{A\tau}\begin{pmatrix} \frac{kT}{C} & 0 \\ 0 & \frac{kT}{J} \end{pmatrix}}$$

3.9.2.

$$\begin{cases} D\dfrac{d\varphi}{dt} = m + H\dot{\theta} \\ \dot{\theta} = -c\varphi \end{cases}$$

$$\begin{cases} \dot{\varphi} = \dfrac{m}{D} - \dfrac{HC}{D}\varphi \\ \dot{\theta} = -C\varphi \end{cases}$$

$$\begin{cases} d\varphi = -\dfrac{HC}{D}\varphi\, dt + \dfrac{1}{D}dw \\ d\theta = -C\varphi\, dt \end{cases} \qquad \alpha = \frac{HC}{D}, \quad \beta = C$$

$$\frac{dP}{dt} = AP + PA^T + R_1$$

$$\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ -\beta & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} -\alpha & -\beta \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{D}r_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} \dot{P}_1 = -\alpha P_1 - \alpha P_1 + \frac{1}{D}r_1 = -2\alpha P_1 + \frac{1}{D}r_1 \\ \dot{P}_2 = -\alpha P_2 - \beta P_1 \\ \dot{P}_2 = -\beta P_1 - \alpha P_2 \\ \dot{P}_3 = -\beta P_2 - \beta P_2 = -2\beta P_2 \end{cases}$$

3.10.1. $dx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x\,dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dv \qquad E(vv) = 1$

$$dy = \begin{bmatrix} 1 & 0 \end{bmatrix} x\,dt + de \qquad E(ee) = r$$

$$\Phi(t_{i+1}, t_i) = e^{A(t_{i+1}-t_i)} = \begin{bmatrix} 1 & t_{i+1}-t_i \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}}$$

$$\Theta(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} C(s)\Phi(s, t_i)\,ds = \int_{t_i}^{t_{i+1}} \begin{bmatrix} 1 & s-t_i \end{bmatrix} ds = \begin{bmatrix} s & \frac{s^2}{2} - s t_i \end{bmatrix}_{t_i}^{t_{i+1}} =$$

$$= \begin{bmatrix} t_{i+1} - t_i & \frac{t_{i+1}^2}{2} - \frac{t_i^2}{2} - t_{i+1}t_i + t_i^2 \end{bmatrix} = \underbrace{\begin{bmatrix} h & \frac{h^2}{2} \end{bmatrix}}$$

$$E\tilde{v}\tilde{v}^T = \tilde{R}_1 = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s) R_1(s) \Phi^T(t_{i+1}, s)\,ds = \int_{t_i}^{t_{i+1}} \begin{pmatrix} 1 & t_{i+1}-s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 0 \\ t_{i+1}-s & 1 \end{pmatrix} ds = \int_{t_i}^{t_{i+1}} \begin{pmatrix} 0 & t_{i+1}-s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_{i+1}-s & 1 \end{pmatrix} ds = \int_{t_i}^{t_{i+1}} \begin{pmatrix} (t_{i+1}-s)^2 & t_{i+1}-s \\ t_{i+1}-s & 1 \end{pmatrix} ds$$

$$= \begin{bmatrix} -\frac{1}{3}(t_{i+1}-s)^3 & -\frac{1}{2}(t_{i+1}-s)^2 \\ -\frac{1}{2}(t_{i+1}-s)^2 & s \end{bmatrix}_{t_i}^{t_{i+1}} = \underbrace{\begin{pmatrix} h^3/3 & h^2/2 \\ h^2/2 & h \end{pmatrix}}$$

$$E\tilde{v}\tilde{e}^T = \tilde{R}_{12} = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s) R_1(s) \Theta^T(t_{i+1}, s)\,ds = \int_{t_i}^{t_{i+1}} \begin{pmatrix} 1 & t_{i+1}-s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot$$

$$\begin{bmatrix} t_{i+1}-s \\ \frac{1}{2}(t_{i+1}-s)^2 \end{bmatrix} ds = \int_{t_i}^{t_{i+1}} \begin{bmatrix} \frac{1}{2}(t_{i+1}-s)^3 \\ \frac{1}{2}(t_{i+1}-s)^2 \end{bmatrix} ds = \begin{bmatrix} -\frac{1}{8}(t_{i+1}-s)^4 \\ -\frac{1}{6}(t_{i+1}-s)^3 \end{bmatrix}_{t_i}^{t_{i+1}} =$$

$$= \underbrace{\begin{bmatrix} h^4/8 \\ h^3/6 \end{bmatrix}}$$

$$E\tilde{e}\tilde{e}^T = \tilde{R}_2 = \int_{t_i}^{t_{i+1}} \left[\Theta(t_{i+1}, s) R_1 \Theta^T(t_{i+1}, s) + R_2(s)\right] ds =$$

$$= \int_{t_i}^{t_{i+1}} \left[\left(t_{i+1}-s \quad \tfrac{1}{2}(t_{i+1}-s)^2\right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_{i+1}-s \\ \tfrac{1}{2}(t_{i+1}-s)^2 \end{pmatrix} + r\right] ds =$$

$$= \int_{t_i}^{t_{i+1}} \left[\left(0 \quad \tfrac{1}{2}(t_{i+1}-s)^2\right) \begin{pmatrix} t_{i+1}-s \\ \tfrac{1}{2}(t_{i+1}-s) \end{pmatrix} + r\right] ds =$$

$$= \int_{t_i}^{t_{i+1}} \left(\tfrac{1}{4}(t_{i+1}-s)^4 + r\right) ds = \left[\tfrac{1}{20}\left[t_{i+1}-s\right]^5 + rs\right]_{t_i}^{t_{i+1}} =$$

$$= \underline{\tfrac{1}{20} h^5 + rh}$$