2.2.1. $\Omega = [0,1]$ $\qquad T = [0,1]$

$$x(t,\omega) = 0 \qquad \forall t, \omega$$

$$y(t,\omega) = \begin{cases} 1 & t = \omega \\ 0 & \text{annars} \end{cases}$$

$$F_{1x}(\xi_1, t_1) = P(x(t_1) \le \xi_1) = \begin{cases} 0 & \xi_1 < 0 \\ 1 & \xi_1 \ge 0 \end{cases}$$

$$F_{2x}(\xi_1, \xi_2; t_1, t_2) = P(x(t_1) \le \xi_1, x(t_2) \le \xi_2) = \begin{cases} 1 & \xi_1, \xi_2 \ge 0 \\ 0 & \text{annars} \end{cases}$$

$$F_{nx}(\xi_1 \dots \xi_n; t_1, \dots, t_n) = \begin{cases} 1 & \xi_1 \dots \xi_n \ge 0 \\ 0 & \text{annars.} \end{cases}$$

$$\underline{P\{\omega : x(t,\omega) < 0.5, \forall t\} = 1}$$

$$F_{1y}(\eta_1, t_1) = P(y(t_1) \le \eta_1) = \begin{cases} 0 & \eta_1 < 0 \\ 1 & \eta_1 \ge 0 \end{cases}$$

$$F_{2y}(\eta_1; \eta_2; t_1, t_2) = P(y(t_1) < \eta_1, y(t_2) < \eta_2) = \begin{cases} 1 & \eta_1 \eta_2 \ge 0 \\ 0 & \text{annars} \end{cases}$$

$$F_{ny}(\eta_1 \dots \eta_n; t_1 \dots t_n) = \begin{cases} 1 & \eta_1 \dots \eta_n \ge 0 \\ 0 & \text{annars} \end{cases}$$

$$\underline{P\{\omega : y(t,\omega) < 0.5, \forall t\} = 0}$$

2.2.2. $x(t) = e(t) + c\,e(t-1)$ $\qquad e(t) \in N(0,1)$

$$\operatorname{cov}[x(t), x(s)] = E(x(t)x(s)) = E\,[e(t) + c\,e(t-1)]\cdot$$

$$[e(s) + c\,e(s-1)] = E[e(t)\,e(s) + c\,e(t-1)\,e(s) +$$

$$+ c\,e(t)\,e(s-1) + c^2 e(t-1)\,e(s-1)]$$

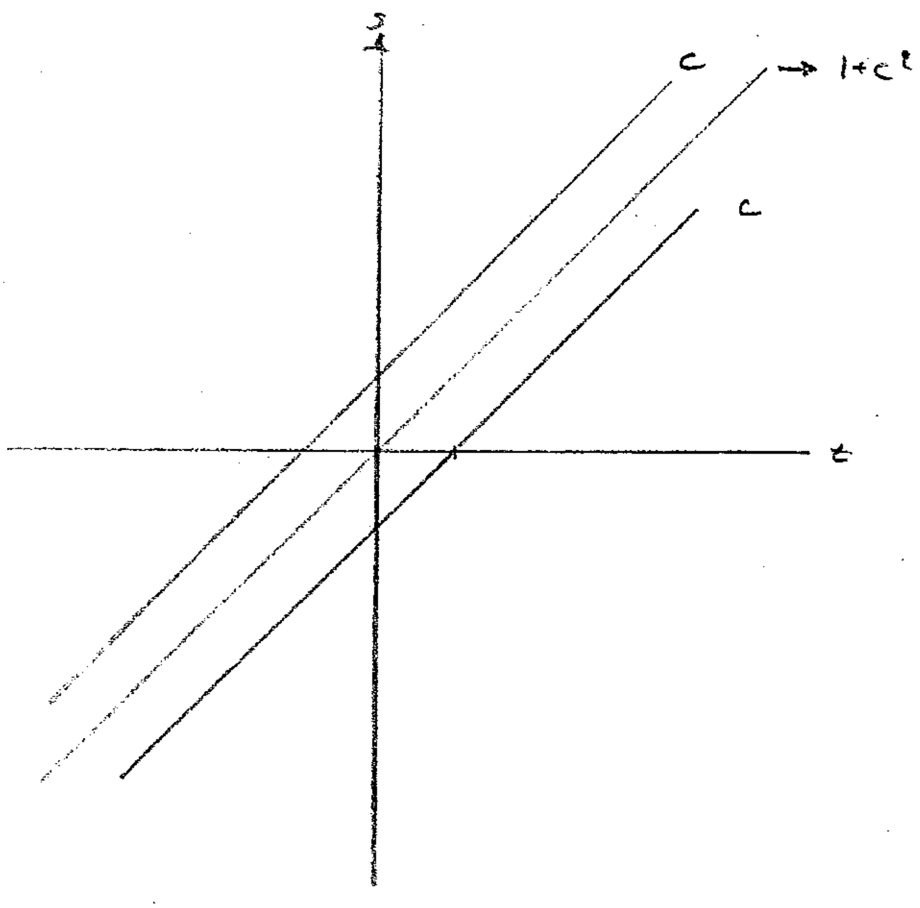
1) $t < s-1$ $\qquad \operatorname{cov} = 0$

2) $t = s-1$ $\qquad \operatorname{cov} = c\,e(s-1)\,e(s-1) = \underline{\underline{c}}$

3) $t = s$ $\qquad \operatorname{cov} = e(s)e(s) + c^2 e(s-1)\,e(s-1) = 1 + c^2$

4) $t = s+1$ $\qquad \operatorname{cov} = c\,e(s)\,e(s) = c$

5) $t > s+1$ $\qquad \operatorname{cov} = 0$



2.2.3. $x(t) + a\,x(t-1) = e(t)$

$|a| < 1$

$e(t) \in N(0,1)$

$$x(t) = -a\,x(t-1) + e(t)$$

$$x(t) = (-a)^{t-t_0} x(t_0) + \sum_{k=0}^{t-t_0-1} (-a)^k e(t-k)$$

$$m(t) = E(x(t)) = (-a)^{t-t_0} x(t_0)$$

$$\operatorname{cov}(x(s), x(t)) = E(x(s) - m(s))(x(t) - m(t)) = E\Big(\sum_{k=0}^{s-t_0-1} (-a)^k e(s-k) \Big) \cdot$$

$$\cdot \Big(\sum_{\ell=0}^{t-t_0-1} (-a)^{\ell} e(t-\ell) \Big) = [\neq 0,\ s-k = t-\ell] =$$

$$= \underline{\sum_{k=0}^{s-t_0-1} (-a)^{2k+t-s}} \qquad (\text{Om } s \leq t\,!)$$

2.3.4. $m_x = m \qquad R_x = R_0 \qquad x \in N(m, R_0)$

$V = x^T S x \qquad S$ symmetrisk.

$$E(V) = E(x^TSx) = E(x-m)^T S (x-m) + E m^T S x + E x^T S m -$$

$$- E m^T S m = E\, tr (x-m)(x-m)^T S + m^T S m = tr\, E (x-m)(x-m)^T S +$$

$$+ m^T S m = \underline{m^T S m + tr\, R_0 S}$$

2.3.5. $x(t) = e(t) + c\,e(t-1)$

$$\left.\begin{array}{l} \underline{\text{Stationär:}}\ m(t) = 0 \\ \quad r(s,t) \text{ beror endast av } s-t \quad (\text{Se } 2.2.2) \end{array}\right\} \Rightarrow \underline{x(t) \text{ stationär}}$$

$\underline{\text{Normal:}}$ $x(t) \in N(0, \sqrt{1+c^2})$

$\underline{\text{Markovprocess:}}$ $P\{x(t) \leq \xi \mid x(t-1), x(t-2) \ldots\} = P(e(t) + c\,e(t-1) \leq \xi \mid \ldots$

$= P\{x(t) \leq \xi \mid x(t-1)\} \Rightarrow \underline{x(t) \text{ är en Markovprocess}}$

$\underline{\text{Ergodisk:}}$ $E(x(t)) = 0$

$$\lim_{T\to\infty} \frac{1}{2T} \sum_{-T}^{T} x(t,\omega) = \lim_{T\to\infty} \frac{1}{2T} \sum_{-T}^{T} e(t,\omega) +$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \sum_{-T}^{T} c\,e(t,\omega) = 0$$

$\Rightarrow$ $\underline{\text{ergodisk}}$

Singulär: $P\{\omega : \alpha x \neq 0\} = P\{\omega : \alpha(e(t) + c e(t-1)) \neq 0\} \neq 0$

$\Rightarrow$ Icke-singulär.

Oberoende inkrement: . .

$$x(t+1) - x(t) = e(t+1) + c e(t) - e(t) - c e(t-1) =$$
$$= e(t+1) + (c-1) e(t) - c e(t-1)$$

$$x(t+2) - x(t+1) = e(t+2) + (c-1) e(t+1) - c e(t)$$

$\Rightarrow$ ej oberoende inkrement

2.3.6. $x(t+1) + a x(t) = e(t) \qquad t = t_0, t_0+1, \ldots$

$|a| < 1 \qquad e(t) \in N(0,1) \qquad x(t_0) \in N(0, \sigma)$

Stationär: $m(t) = 0$

$$r(s,t) = (-a)^{s-t} \left(\sum_{i=0}^{t-t_0-1} (a^2)^i + (a^2)^{t-t_0} \sigma^2 \right)$$ beror ej endast av $s-t$

$\Rightarrow$ processen ej stationär

Normal: $x(t)$ = summa av normalprocesser $\Rightarrow$ normalprocess

Markovprocess: $P(x(t) \leq \xi \mid x(t-1), \ldots x(t_0)) = P(-a x(t-1) + e(t-1) \leq \xi \mid \ldots) =$

$= P(-a x(t-1) + e(t) \leq \xi \mid x(t-1)) \Rightarrow$ Markovprocess

Ergodisk: Processen ej stationär $\Rightarrow$ Ej ergodisk

Singulär: $P\{\omega : \alpha x \neq 0\} = P\{\omega : \alpha(-a x(t-1) + e(t-1)) \neq 0\} \neq 0$

$\Rightarrow$ Processen ej singulär

Oberoende inkrement: $x(t+1) - x(t) = -ax(t) + e(t) + ax(t-1) - e(t-1)$

$$x(t+2) - x(t+1) = -ax(t+1) + e(t+1) + ax(t) - e(t)$$

$\Rightarrow$ Ej oberoende inkrement

2.3.7. $x(t+1) + ax(t) = e(t) \qquad t = t_0, t_0+1, \ldots$

$|a| < 1$

$e(t) \in N(0,1)$

$x(t_0) \in N(0, \sigma)$

$$z = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \qquad E(zz^T) = \begin{pmatrix} 1 & \rho \\ \rho & \sigma^2 \end{pmatrix}$$

Vi har fortfarande en Markovprocess, eftersom vi vid prediktion av $x(t+1)$ får maximal information om vi endast känner $x(t)$.

2.3.8. $\frac{dx}{dt} = 0 \qquad 0 \le t < \infty$

$x(0) \in N(0,1)$

$x(t) = x(0).$

$E(x(t)) = E(x(0)) = 0$

$\lim_{T\to\infty} \frac{1}{2T} \int_0^T x(t,\omega)\,dt \neq 0 \quad , x(0) \neq 0.$

$\Rightarrow$ <u>Processen är ej ergodisk</u>

$\underline{\frac{x(t+h) - x(t)}{h}} = 0$

$\underline{x(t+h) = x(t)}$

2.3.9. $\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$

$\mathrm{cov}[x(0), x(0)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad x(0) \in N(0,1)$

$\phi(t,0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$m(t) = \phi(t,0)\,x(0) = 0$

Processen är stationär.

$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t,\omega)\,dt = 0$

$\Rightarrow$ <u>Processen är ergodisk</u>

$$x(t+h) = \phi(t+h, t_0)\, x(t_0)$$

$$x_1(t_0) = x_1^0$$

$$x_1(t) = \cos(t-t_0)\, x_1^0 + \sin(t-t_0)\, x_2^0$$

$$x_2^0 = \frac{x_1(t) - \cos(t-t_0) x_1(t_0)}{\sin(t-t_0)}$$

$$x(t+h) = \begin{bmatrix} \cos(t+h-t_0) & \sin(t+h-t_0) \\ -\sin(t+h-t_0) & \cos(t+h-t_0) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ \dfrac{x_1(t) - \cos(t-t_0) x_1(t_0)}{\sin(t-t_0)} \end{bmatrix}$$

2.4.1. Villkor: 1) $|r(\tau)| \le r(0)$

2) $r(0) \ge 0$

3) $|r(\tau)| = r(0),\ \tau \ne 0 \Rightarrow r$ är periodisk

4) $r(\tau) = r(-\tau)$

5) Om $r(\tau)$ kont i origo $\Rightarrow$ kont. överallt.

a) $r(\tau) =$ konstant.

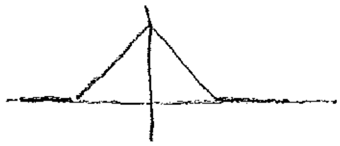
<u>Kovariansfkn om konstanten ≥ 0.</u>

b) $r(\tau) = \cos \tau$. <u>Kovariansfkn</u>

c) $r(\tau) = \begin{cases} 1 & |\tau| < 1 \\ 0 & |\tau| \ge 1 \end{cases}$

<u>Ej kovariansfkn</u> 5) gäller ej.

d) $r(\tau) = \begin{cases} 1 - |\tau| & |\tau| < 1 \\ 0 & |\tau| \ge 1 \end{cases}$



<u>Kovariansfkn</u>

e) $r(\tau) = \dfrac{1}{1 + 2\xi|\tau| + \tau^2}$

<u>Kovariansfkn</u>

f. $r(\tau) = \begin{cases} 2 & \tau = 0 \\ e^{-|\tau|} & \tau \ne 0 \end{cases}$

<u>Kovariansfkn</u>

2.4.2. $x(t) + a x(t-1) = e(t) + c\, e(t-1)$

$|a| < 1$

$e(t) \in N(0,1)$

$$x(t) = -a x(t-1) + e(t) + c\, e(t-1) = (-a)^t x(0) + \sum_{k=0}^{t-1} (-a)^k \left[e(t-k) + c e(t- \right.$$

$$m(t) = (-a)^t x(0)$$

$$\operatorname{cov}[x(s), x(t)] = E\left[x(s) - m(s)\right]\left[x(t) - m(t)\right] =$$

$$= E\left[\sum_{k=0}^{s-1} (-a)^k \left[e(s-k) + c\, e(s-k-1)\right]\right]\left[\sum_{\ell=0}^{t-1} (-a)^\ell \left[e(t-\ell) + c\, e(t-\ell-1)\right]\right]$$

Arbetsamma räkningar följer!

2.4.5. $\underline{\operatorname{cov}[Ax+a, By+b]} = E[(Ax+a) - E(Ax+a)][(By+b) - E(By+b)]^T =$

$$= E[Ax + a - AE(x) - a][By + b - BE(y) - b]^T =$$

$$= E[Ax - AE(x)][By - BE(y)]^T = E\{A[x - E(x)][y - E(y)]^T B^T$$

$$= A E[x - E(x)][y - E(y)]^T B^T = \underline{A \operatorname{cov}[x, y] B^T}$$

2.5.1. a) $r(\tau) = e^{-a|\tau|}$

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau - a|\tau|}\, d\tau =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-i\omega\tau + a\tau}\, d\tau + \frac{1}{2\pi}\int_{0}^{\infty} e^{-i\omega\tau - a\tau}\, d\tau =$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-i\omega}\, e^{-i\omega\tau + a\tau}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[\frac{-1}{a+i\omega}\, e^{-i\omega\tau - a\tau}\right]_{0}^{\infty} =$$

$$= \frac{1}{2\pi}\left(\frac{1}{a-i\omega} + \frac{1}{a+i\omega}\right) = \underline{\underline{\frac{a}{\pi(a^2+\omega^2)}}}$$

b) $r(\tau) = e^{-\alpha^2\tau^2}$

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\tau^2 - i\omega\tau}\, d\tau$$

$$\alpha^2\tau^2 + i\omega\tau = \alpha^2\tau^2 + i\omega\tau - \frac{\omega^2}{4\alpha^2} + \frac{\omega^2}{4\alpha^2} = \left(\alpha\tau + \frac{i\omega}{2\alpha}\right)^2 + \frac{\omega^2}{4\alpha^2}$$

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\alpha\tau + \frac{i\omega}{2\alpha}\right)^2} \cdot e^{-\frac{\omega^2}{4\alpha^2}}\, d\tau = \frac{1}{2\pi} e^{-\frac{\omega^2}{4\alpha^2}} \int_{-\infty}^{\infty} e^{-x^2}\, d\tau$$

$$\frac{dx}{d\tau} = \alpha \Rightarrow d\tau = \frac{1}{\alpha}\, dx$$

$$\phi(\omega) = \frac{1}{2\pi\alpha}\, e^{-\frac{\omega^2}{4\alpha^2}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = \underline{\underline{\frac{e^{-\frac{\omega^2}{4\alpha^2}}}{2\alpha\sqrt{\pi}}}}$$

c) $r(\tau) = A + B\cos\omega\tau$

$$\Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A e^{-i\omega\tau} d\tau + \frac{B}{2\pi}\int_{-\infty}^{\infty} \cos\omega_0\tau \, e^{-i\omega\tau} d\tau = A\delta(\omega) +$$

$$+ \frac{B}{4\pi}\int_{-\infty}^{\infty}\left(e^{i\omega_0\tau} \cdot e^{-i\omega\tau} + e^{-i\omega_0\tau} \cdot e^{-i\omega\tau}\right) d\tau = \underline{A\delta(\omega) +}$$

$$\underline{+ \frac{B}{2}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)}$$

d) $r(\tau) = e^{-\alpha|\tau|}\cos\beta\tau$

$$\Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\tau| - i\omega\tau} \cdot \frac{1}{2}\left(e^{i\beta\tau} + e^{-i\beta\tau}\right) d\tau =$$

$$= \frac{1}{4\pi}\int_{-\infty}^{0} e^{+\alpha\tau - i\omega\tau + i\beta\tau} d\tau + \frac{1}{4\pi}\int_{-\infty}^{0} e^{+\alpha\tau - i\omega\tau - i\beta\tau} d\tau +$$

$$+ \frac{1}{4\pi}\int_{0}^{\infty} e^{-\alpha\tau - i\omega\tau + i\beta\tau} d\tau + \frac{1}{4\pi}\int_{0}^{\infty} e^{-\alpha\tau - i\omega\tau - i\beta\tau} d\tau =$$

$$= \frac{1}{4\pi}\left[\frac{1}{\alpha - i(\omega-\beta)} e^{\cdots} + \frac{1}{\alpha - i(\omega+\beta)} e^{\cdots}\right]_{-\infty}^{0} +$$

$$+ \frac{1}{4\pi}\left[\frac{-1}{\alpha + i(\omega-\beta)} e^{\cdots} + \frac{-1}{\alpha + i(\omega+\beta)} e^{\cdots}\right]_{0}^{\infty} =$$

$$= \frac{1}{4\pi}\left(\frac{1}{\alpha - i(\omega-\beta)} + \frac{1}{\alpha - i(\omega+\beta)} + \frac{1}{\alpha + i(\omega-\beta)} + \frac{1}{\alpha + i(\omega+\beta)}\right) =$$

$$= \underline{\frac{1}{4\pi}\left(\frac{2\alpha}{\alpha^2 + (\omega-\beta)^2} + \frac{2\alpha}{\alpha^2 + (\omega+\beta)^2}\right)}$$

2.5.2. a) $x(t) = e(t) + c\,e(t-1)$

$$r(\tau) = \begin{cases} 1+c^2 & \tau = 0 \\ c & \tau = 1 \\ 0 & \tau > 1 \end{cases}$$

$$\phi(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r(n) e^{-in\omega} = \frac{1}{2\pi}\left(1 + c^2 + c e^{-i\omega} + c e^{i\omega}\right) =$$

$$= \underline{\frac{1}{2\pi}\left(1 + c^2 + 2c\cos\omega\right)}$$

2.6.1. a) $r(\tau) = e^{-\alpha|\tau|}$

$m(t) =$ konstant $\Rightarrow$ kontinuerlig

$r(0) = 1$ kontinuerlig

} Processen kontinuerlig

b) $r(\tau) = \begin{cases} 2 & \tau = 0 \\ e^{-|\tau|} & \tau \neq 0 \end{cases}$

$r(\tau)$ är ej kontinuerlig i origo $\Rightarrow$ Processen ej kontinuerlig

2.6.2. a) $r(\tau) = e^{-\alpha|\tau|}$

$m(t) =$ konstant $\Rightarrow$ differentierbar

$r'(\tau) = \begin{cases} -\alpha\, e^{-\alpha\tau} & \tau > 0 \\ \alpha\, e^{\alpha\tau} & \tau < 0 \end{cases}$

$r(\tau)$ ej två ggr deriverbar i origo $\Rightarrow$ Processen är ej differentierbar

b) $r(\tau) = \dfrac{\alpha}{\alpha^2 + \tau^2}$

$r'(\tau) = \dfrac{-2\tau\alpha}{(\alpha^2 + \tau^2)^2}$

$r''(\tau) = \dfrac{-2\alpha}{(\alpha^2+\tau^2)^2} - \dfrac{2\tau\alpha \cdot 2\tau}{(\alpha^2-\tau^2)^3}$

$r''(\tau)$ definierad i origo $\Rightarrow$ Processen är differentierbar

2.6.4. $r(\tau) = (1+|\tau|)e^{-|\tau|} = \begin{cases} (1+\tau)e^{-\tau} & \tau > 0 \\ (1-\tau)e^{\tau} & \tau < 0 \end{cases}$

$$r'(\tau) = \begin{cases} e^{-\tau} - (1+\tau)e^{-\tau} = -\tau e^{-\tau} & \tau > 0 \\ -e^{\tau} + (1-\tau)e^{\tau} = -\tau e^{\tau} & \tau < 0 \end{cases} \qquad r'(0) = 0$$

$$r''(\tau) = \begin{cases} -e^{-\tau} + \tau e^{-\tau} & \tau > 0 \\ -e^{\tau} - \tau e^{\tau} & \tau < 0 \end{cases} \qquad r''(0) = -1$$

$m(t) = \text{konstant}$

$\therefore$ Processen är differentierbar

$$\operatorname{cov}\left(\frac{d}{ds}x(s), x(t)\right) = \frac{\partial}{\partial s} r(s,t) = \frac{\partial}{\partial s} \begin{cases} (1+s-t)e^{-s+t}, & s > t \\ (1-s+t)e^{s-t}, & s < t \end{cases} =$$

$$= \begin{cases} e^{-s+t} - (1+s-t)e^{-s+t}, & s > t \\ -e^{s-t} + (1-s+t)e^{s-t}, & s < t \end{cases} = \begin{cases} -(s-t)e^{-s+t} & s > t \\ -(s-t)e^{s-t} & t > s \end{cases}$$

$s = t \Rightarrow \operatorname{cov}\left(\frac{d}{dt}x(t), x(t)\right) = 0 \Rightarrow x$ och $\frac{dx}{dt}$

är okorrelerade. Om man kan visa att $\frac{dx}{dt}$ är en normalprocess är $\frac{dx}{dt}$ och x oberoende. Se SSP sid. 4.17!