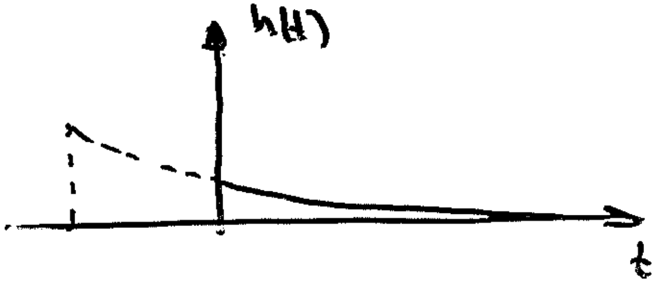
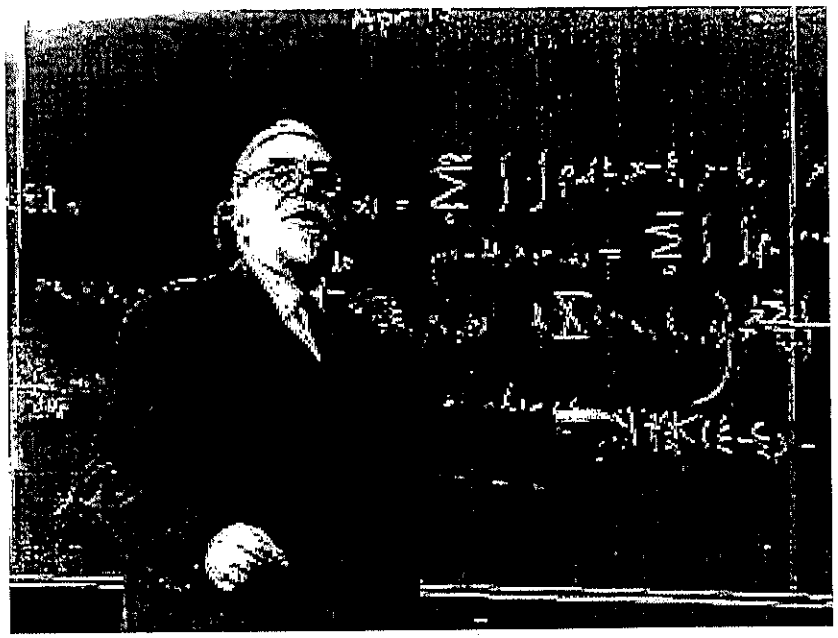
Dagens idé:

Hur fås ett optimalt kausalt system?



Norbert Wiener
1894 - 1964



Wienerteori

1. Inledning
2. Problemformulering
3. Optimala filter
4. Svårigheter
5. Jämförelse med Kalmanfilter
6. Polynommetoder
7. Sammanfattning

Optimala filter

Wiener, N. 1942 MIT

"The Yellow Peril"

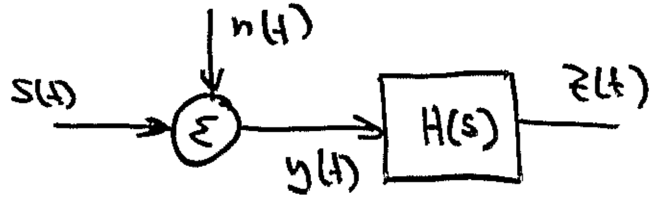
1949

Tidskontinuerligt

Kolmogorov, A.N. 1941

Tidsdiskret

PROBLEM



Bestäm $H(s)$ så att

$$\mathcal{E} = E\left[z(t) - s(t+\eta) \right]^2 \quad \text{minimeras}$$

$\eta = 0$ Filtrering

$\eta > 0$ Prediktion

$\eta < 0$ Utjämning

OBS Stationaritet

Använd $y(\tau) \quad -\infty < \tau < t$

Förutsättningar

- Tidsinvarianta system
- Kända spektraltätheter och kovariansfunktioner
- Rationella spektraltätheter
- Ergodicitet
- Använd y $(-\infty, t]$

Ansätser:

1) Variationskalkyl $\Leftarrow$

2) Polynomberäkningar

För- och nackdelar

"Lösning"

$$\mathcal{E} = E\left[s(t+\eta) - \int_0^\infty h(\tau)y(t-\tau)d\tau \right]^2$$

$$= R_s(0) - 2\int_0^\infty h(\tau)R_{sy}(\eta+\tau)d\tau$$

$$+ \int_0^\infty\!\!\int h(\tau)h(\mu)R_y(\tau-\mu)d\tau\, d\mu$$

Vilket $h(t)$ sådant att $h(t) = 0 \quad t<0$ ger minst förlust?

Variationskalkyl

- Antag $h(t)$ optimala filtret
- Ersätt $h(t)$ med

  $h(t) + \varepsilon g(t)$
- Titta på förändringen i $\mathcal{E}$

$$\mathcal{E} + \Delta\mathcal{E} = \mathcal{E}$$

$$-2\varepsilon\left[\int_0^\infty g(\tau)\left[R_{sy}(\eta+\tau) - \int_0^\infty h(\mu)R_y(\tau-\mu)\,d\mu\right]d\tau\right]$$

$$+\varepsilon^2 \iint_0^\infty g(\tau)g(\mu)R_y(\tau-\mu)\,d\tau\,d\mu$$

$$= \mathcal{E} - 2\varepsilon\,\Delta\mathcal{E}_1 + \varepsilon^2\,\underbrace{\Delta\mathcal{E}_2}_{\geq 0}$$

$$\Rightarrow \Delta\mathcal{E} < 0 \quad \text{om} \quad \Delta\mathcal{E}_1 \neq 0$$

$$\boxed{R_{sy}(\tau+\eta) = \int_0^\infty h(\mu)R_y(\tau-\mu)\,d\mu \qquad \tau \geq 0}$$

Wiener-Hopf's integralekv

Nödv. och tillr. villkor

Specialfall:
Bortse från realiserbarheten

Fourietransformera W-H

$\Rightarrow$

$$e^{s\eta}\,\phi_{sy}(\omega) = H(s)\,\phi_y(\omega)$$

$$H(s) = \frac{\phi_{sy}(\omega)}{\phi_y(\omega)}\,e^{s\eta}$$

<u>Två problem</u>

- Poler i HHP
- $e^{s\eta}$ då $\eta > 0$ (prediktion)

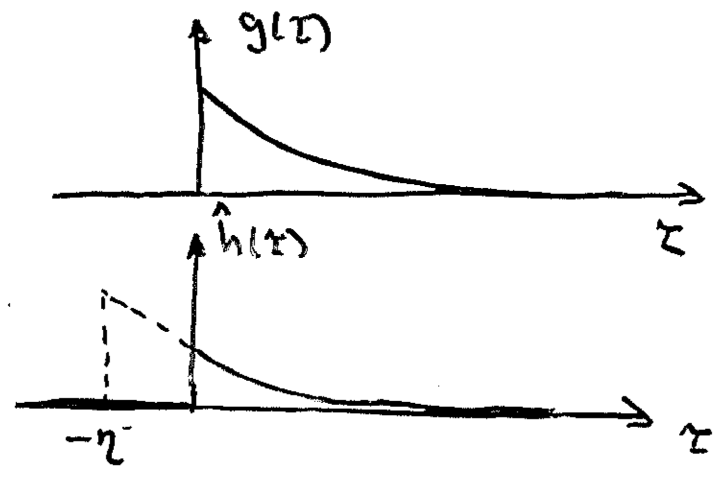
Specialfall:
Ren prediktion $\eta = 0$

$y(t) = s(t)$

* Innovationsbeskr. av $s(t)$

$$e(t) \rightarrow \boxed{G(s)} \xrightarrow{s(t)} \underbrace{\boxed{G^{-1}(s)} \xrightarrow{e} \boxed{\hat{H}(s)}}_{H(s) = G^{-1}(s)\hat{H}(s)} \rightarrow$$

$$\hat{h}(\tau) = \begin{cases} 0 & \tau < 0 \\ g(\tau+\eta) & \tau \geq 0 \end{cases}$$



TVÅ TRICK

1) Spektralfaktorera

$$\Phi_y(\omega) = G(i\omega)G(-i\omega) = \underset{g(t)}{G(i\omega)}\,\underset{g'(t)}{G^*(i\omega)}$$

2) $$\Phi_{sy}(\omega) = A(\omega)G^*(i\omega)$$

1) $\Rightarrow$ $$R_y(\tau) = \int_{-\infty}^{0} g(\tau-\mu)g'(\mu)\,d\mu$$

2) $\Rightarrow$ $$R_{sy}(\tau) = \int_{-\infty}^{0} a(\tau-\mu)g'(\mu)\,d\mu$$

OBS Övre gräns 0 för

$g'(\tau)=0 \quad \tau>0$

Sätt in i W-H

⋮

$H(i\omega) =$

$$= \frac{1}{G(i\omega)}\cdot\underbrace{\frac{1}{2\pi}\int_0^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}\,\Phi_{sy}(\omega)}{G^*(i\omega)}\,d\omega\,d\tau}_{\left[\frac{\Phi_{sy}(\omega)}{G^*(i\omega)}\right]_{pr}}$$

pr — eller +

Realiserbara delen

Jfr

$$H(i\omega) = \frac{1}{G(i\omega)}\cdot\frac{\Phi_{sy}(\omega)}{G^*(i\omega)}$$

HUR BESTÄMS $[\ \ ]_+$?

$$\frac{\Phi_{sy}(\omega)}{G^*(i\omega)} = \underbrace{\sum_{stab} \frac{\beta_i}{i\omega+\alpha_i}}_{\left[\frac{\Phi_{sy}(\omega)}{G^*(i\omega)}\right]_+} + \sum_{instab} \frac{\gamma_i}{i\omega-\alpha_i}$$

EXEMPEL – FILTRERING

$$\Phi_s = \frac{1}{1+\omega^2} \qquad \Phi_n = a^2$$

s, n okorr

$y = s + n$

$$\Phi_y = \Phi_s + \Phi_n$$

$$= \frac{i\omega a + \sqrt{1+a^2}}{i\omega + 1} \cdot \frac{-i\omega a + \sqrt{1+a^2}}{-i\omega + 1} = GG^*$$

$$\Phi_{ys} = \Phi_s$$

$$\frac{\Phi_{ys}}{G^*} = \frac{1}{1+\omega^2} \cdot \frac{-i\omega + 1}{(-ia\omega + \sqrt{1+a^2})}$$

$$= \frac{1}{(i\omega+1)(-ia\omega+\sqrt{1+a^2})}$$

$$= \underbrace{\frac{1}{a+\sqrt{1+a^2}} \cdot \frac{1}{i\omega+1}}_{[\ \ ]_+} + \frac{\sim\sim}{-ia\omega+\sqrt{1+a^2}}$$

$$H(s) = \frac{s+1}{sa+\sqrt{1+a^2}} \cdot \frac{1}{s+1} \cdot \frac{1}{a+\sqrt{1+a^2}}$$

$$\boxed{H(s) = \frac{1}{a+\sqrt{1+a^2}} \cdot \frac{1}{a} \cdot \frac{1}{s+\sqrt{1+1/a^2}}}$$

KALMANFILTER

$$\dot{x} = -x + e \qquad e \in N(0,1)$$
$$y = x + v \qquad v \in N(0,a)$$

$$\dot{P} = -P - P - \frac{P^2}{a^2} + 1$$

$$\dot{P} = 0 \Rightarrow \quad P = a(\sqrt{1+a^2} - a)$$

$$K = \frac{\sqrt{1+a^2}}{a} - 1$$

$$(s+1+K)\hat{X}(s) = K\,Y(s)$$

$$H(s) = \frac{\sqrt{1+a^2} - a}{a} \cdot \frac{1}{s + \sqrt{1+1/a^2}}$$

$$\boxed{H(s) = \frac{1}{\sqrt{1+a^2} + a} \cdot \frac{1}{a} \cdot \frac{1}{s + \sqrt{1+1/a^2}}}$$

EXEMPEL – PREDIKTION $\begin{cases} n = 0 \\ s(t+\eta) \end{cases}$

$$\Phi_s(\omega) = \frac{1}{(1+\omega^2)^2} = \underset{G}{\frac{1}{(1+i\omega)^2}} \cdot \underset{G^*}{\frac{1}{(1-i\omega)^2}}$$

$$\Phi_{\hat{s}y} = \frac{e^{i\omega\eta}}{(1+\omega^2)^2}$$

$$H(i\omega) = \frac{1}{G(i\omega)} \int_0^\infty e^{-i\omega\tau} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(\tau+\eta)} \cdot \frac{1}{(1+i\omega)^2} \, d\omega}_{(\tau+\eta)\,e^{-(\tau+\eta)}} \, d\tau$$

$$H(i\omega) = (1+i\omega)^2 \, e^{-\eta}\left[\frac{1}{(1+i\omega)^2} + \eta \frac{1}{1+i\omega}\right]$$
$$= e^{-\eta}\left[1 + \eta + \eta i\omega\right]$$

$$H(s) = \eta e^{-\eta}\left[s + \frac{1+\eta}{\eta}\right]$$

OBS Derivering

POLYNOMFORMULERING

- Bra sätt att beräkna $[\ ]_+$?
- Polynomidentiteter bra för I/O formuleringar

Kučera ~1978

Ahlén - Sternad 1990

Ref: T. Söderström (2002): Discrete time stochastic systems, Springer Verlag

Diskret tid

$$\boxed{G(z) = \left[\frac{\Phi_{sy}(z)}{\lambda^2 H^*(z)}\right]_+ \frac{1}{H(z)}}$$

$y = He$

Söderströms beteckningar

$A(q)y(t) = C(q)e(t) \quad e \in N(0,\lambda)$

$$\Phi_y(z) = G(z)\Phi_u(z)G^*(z^{-*})$$

$$= G(z)G(z^{-1})\Phi_u(z)$$

(reellt, skalärt)

$$C(z) = z^n + c_1 z^{n-1} + \dots + c_n$$

$$C^*(z^{-*}) = C(z^{-1})$$

$$= z^{-n} + c_1 z^{-n+1} + \dots + c_n$$

$$C^*(z) = [C(z)]^*$$