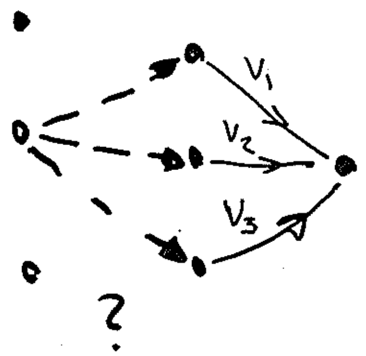
Dagens idé:



Dynamisk programmering

LQG

- Formulering
- Förarbeten
- Fullst. tillståndsinfo. tidsdiskr
- Ofullst — " —
- Separation
- Tidskont. fallet

Problem

$$\begin{cases} x(t+1) = \phi x(t) + \Gamma u(t) + v(t) & R_1 \\ y(t) = \theta x(t) + e(t) & R_{12} = 0 \\ & R_2 \end{cases}$$

m, R_0 givet

Minimera

$$J = E\ell = E\Big\{x^T(N) Q_0 x(N) + \sum_{t=t_0}^{N-1} \big(\underbrace{x^T(t) Q_1 x(t)}_{>0} + \underbrace{u^T(t) Q_2 u(t)}_{\geq 0}\big)\Big\}$$

Om Q_{12}

$$2x^T(t) Q_{12} u(t)$$

Inför $\tilde{u} = u + M^T x$

$$M = Q_{12} Q_2^{-1}$$

$$x(t+1) = \tilde{\phi} x(t) + \tilde{\Gamma} u(t) + v(t)$$

$$y(t) = \theta x(t) + e(t)$$

$$\tilde{\phi} = \phi - \Gamma M^T$$

$$\tilde{Q}_1 = Q_1 - Q_{12} Q_2^{-1} Q_{12}^T$$

Fria parametrar

$Q_1 > 0$ $\quad Q_{12}$ $\quad Q_2 \geq 0$

Tidskont. förlustfunktion

$$J = E\Big\{\int_0^{Nh} [x^T Q_{1c} x + 2x^T Q_{12c} u + u^T Q_{2c} u]\, dt + x^T(Nh) Q_{0c} x(Nh)\Big\}$$

Integrera över ett samplings-intervall

Styrlagar

- $y(t) = x(t)$ fullst tillst. info $u(x, t)$
- Ofullst. tillst. info. $u(\mathcal{Y}_{t-1})$, $u(\mathcal{Y}_t)$

Några lemma

Lemma 3.1

$$\min_{u(x,y)} E\,\ell(x,y,u) = E \min_{u} \ell(x,y,u)$$

Lemma 3.2

$$\min_{u(y)} E\,\ell(x,y,u) = E_y\{\min E\{\ell(x,y,u)\,|\,y\}\}$$

$$\min_{u(y)} E\ell \geq \min_{u(x,y)} E\ell$$

Lemma 3.3

$$E\,x^T S x = m^T S m + tr(SR)$$

$$x \in N(m, R)$$

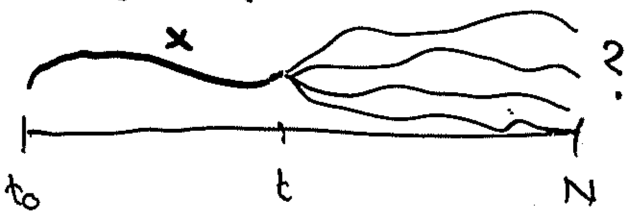
$$E\,x^T S x = E(x-m)^T S(x-m) + m^T S m$$

$$(x-m)^T S(x-m) = tr(\;\;)^T S(\;\;) = tr\,S(x-m)(x-m)^T$$

$$E(x-m)^T S(x-m) \;\; = tr(SR)$$

Fullständig tillst. info

Dynamisk programmering (Bellman)
Optimalitetsprincipen
Bygger på Hamilton-Jacobi formul.



$u(t) = u(x(t_0) \ldots x(t))$

Hur påverkar $u(t)$ framtida $x(t)$?

Förlustfunktionen

$$J = \overset{\min}{E}\left\{ \sum_{t_0}^{t-1} x^T Q_1 x + u^T Q_2 u \right\}$$

oberoende av framtida u

$$+ \underbrace{\overset{\min}{E}\left\{ x^T(N) Q_N x(N) + \sum_{t}^{N-1} x^T Q_1 x + u^T Q_2 u \right\}}_{J_f}$$

Lemma 3.1

$$J_f = \underbrace{E \min_{u(t) \dots u(N-1)} \underset{x(t+1) \dots x(N)}{E} \left\{ x^T(N) Q_0 x(N) + \sum_{t}^{N-1} x^T Q_1 x + u^T Q_2 u \,\middle|\, x(t) \right\}}_{V(x,t)}$$

Bellman ekv

$$V(x(t),t) = \min_{u(t)} E\left[x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t)\right.$$

$$+ \min_{u(t+1)} E\left[x^T(t+1) Q_1 x(t+1) + u^T(t+1) Q_2 u(t+1)\right.$$

$$+ \min_{u(t+2)} E\left[\dots \,|\, x(t+2)\right] \,|\, x(t+1)\Big] \,\Big|\, x(t)\Big]$$

$\vdots$

$$V(x(t),t) = \min_{u(t)} \Big[x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t) + E\left\{ V(x(t+1), t+1) \,|\, x(t) \right\} \Big] \quad (*)$$

Initialvärde

$$V(x,N) = \min_{u(N)} E\left[x^T(N) Q_0 x(N) \,|\, x \right] = x^T Q_0 x$$

Kan visa att lösningen till $(*)$ kan skrivas på formen

$$V(x,t) = x^T \underset{>0}{S(t)} x + s(t)$$

Använd induktion

Resultat

$$u(t) = -L(t)x(t)$$

$$S(t) = \Phi^T S(t+1)\Phi + Q_1 - L^T(Q_2 + \Gamma^T S(t+1)\Gamma)^{-1}L$$

$$S(N) = Q_0$$

$$\min E\{\ \} = E[V(x,t_0)|x]$$

$$= E[x^T(t_0)S(t_0)x(t_0) + s(t_0)]$$

$$= m^T S(t_0) m + \operatorname{tr} S(t_0) R_0 \quad \text{Initialv.}$$

$$+ \sum_{t=t_0}^{N-1} \operatorname{tr} R_1 S(t+1) \qquad \text{Brus } t_0 .. N-1$$

OFULLST. TILLSTÅNDSINFO.

$$u(t) = f(y_{t-1})$$

$$E\left[\sum_{t_0}^{t-1} x^T Q_1 x + u^T Q_2 u\right]$$

$$+ E\left[x^T(N) Q_0 x(N) + \sum_{t}^{N-1} x^T Q_1 x + u^T Q_2 u\right]$$

$$\min_{u(t)} E\left[x^T(N) Q_0 x(N) + \sum_{s=t}^{N-1} x^T(s) Q_1 x(s) + u^T(s) Q_2 u(s) \,\middle|\, y_{t-1}\right]$$

Samma som tidigare

$$V(y_{t-1},t) = \min_u E\left[x^T Q_1 x + u^T Q_2 u + V(y_t, t+1) \,\middle|\, y_{t-1}\right]$$

Problem med y_t

Inför

$$W(\hat{x}(t),t) = V(y_{t-1},t)$$

$$W(\hat{x}(t),t) = \min_u E\Big[x^T Q_1 x + u^T Q_2 u + W(\hat{x}(t+1),t+1) \,\Big|\, \hat{x}(t)\Big]$$

<u>OBS</u> $\hat{x}(t)$ vet vi hur vi kan få

Blir samma räkningar som tidigare men lite extra termer

$$\underbrace{m^T S(t_0) m + \operatorname{tr} R_0 S(t_0)}_{\text{I.C.}} + \underbrace{\sum_{t_0}^{N-1} \operatorname{tr} R_1 S(t+1)}_{v(t)}$$
$$+ \underbrace{\sum_{t_0}^{N-1} \operatorname{tr} P(t) L^T(t) \Gamma^T S(t+1) \phi}_{e(t)}$$

$$S = S(Q_0, Q_1, Q_2)$$

$$P = P(R_0, R_1, R_2)$$

$$u(t) = -L(t)\hat{x}(t|t-1)$$

<u>SEPARATION</u>

$\hat{x}(t|t)$?

<u>EGENSKAPER</u>

$$x(t+1) = (\phi - \Gamma L)x(t) + \Gamma L \tilde{x}(t) + v$$

$$\tilde{x}(t+1) = (\phi - K\theta)\tilde{x}(t) + v(t) - K e(t)$$

$$\begin{bmatrix} x(t+1) \\ \tilde{x}(t+1) \end{bmatrix} = \begin{bmatrix} \phi - \Gamma L & \Gamma L \\ 0 & \phi - K\theta \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \dots$$

Asymptotiska resultat:

$\phi - \Gamma L$
$\phi - K\theta$ asympt. stab. matriser

Tidskontinuerliga fallet

$$\begin{cases} dx = Ax\,dt + Bu\,dt + dv \\ dy = Cx\,dt + de \end{cases}$$

Förlustfunktion

$$E\left\{ x^T(t_1) Q_0 x(t_1) + \int_{t_0}^{t_1} (x^T Q_1 x + u^T Q_2 u)\,dt \right\}$$

1. Deterministiska fallet

$$dv = de = R_1 = R_2 = 0$$

$$J \geq x^T(t_0) S(t_0) x(t_0)$$

$$u = -Lx = -Q_2^{-1} B^T S x$$

$$EJ = m^T S(t_0) m + \mathrm{tr}\, S(t_0) R_0$$

2. Fullständig tillståndsinformation

$$u = -Lx$$

$$EJ = m^T S(t_0) m + \mathrm{tr}\, S(t_0) R_0 + \int_{t_0}^{t_1} \mathrm{tr}(R_1 S)\,dt$$

3. Ofullst. tillståndsinformation

$$u = -L\hat{x}$$

$$EJ = m^T S(t_0) m + \mathrm{tr}\, S(t_0) R_2 + \int_{t_0}^{t_1} \mathrm{tr}(R_1 S)\,dt + \int_{t_0}^{t_1} \mathrm{tr}(L^T Q_2 L P)\,dt$$

Lemma 7.1

Antag

$$-\frac{dS}{dt} = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S$$

$$S(t_1) = Q_0$$

har lösning ≥ 0 i $[t_0, t_1]$ då

$$J = x^T(t_1) Q_0 x(t_1) + \int_{t_0}^{t_1} [x^T Q_1 x + u^T Q_2 u]\, dt$$

$$= x^T(t_0) S(t_0) x(t_0)$$

$$+ \int (u + Q_2^{-1} B^T S x)^T Q_2 (u + Q_2^{-1} B^T S x)\, dt$$

$$+ \int \mathrm{tr}\, R_1 S\, dt + \int dv^T S x + \int x^T S\, dv$$

Bevis

$$x^T(t_1) Q_0 x(t_1) = x^T(t_1) S(t_1) x(t_1)$$

$$= x^T(t_0) S(t_0) x(t_0) + \int_{t_0}^{t} d(x^T S x)$$

$$d(x^T S x) = dx^T S x + x^T S dx + x^T \frac{dS}{dt} x\, dt + (\mathrm{tr}\, SR_1)\, dt$$

Slutna systemet

$$\begin{bmatrix} dx \\ d\tilde{x} \end{bmatrix} = \begin{bmatrix} A-BL & BL \\ 0 & A-KC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} dt + \begin{bmatrix} dv \\ dv - K\, de \end{bmatrix}$$

Kräv 1) Stabiliserbarhet $[A, B]$

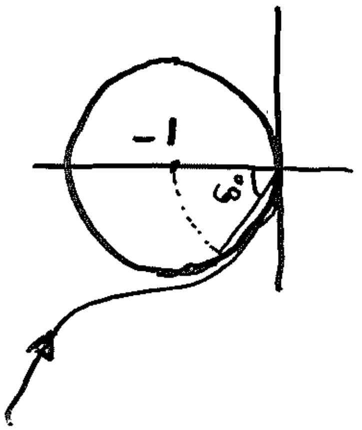
2) Detekterbarhet $[A, D]$ $DD^T = Q_1$

$\Rightarrow$ $S(\infty)$ existerar entydigt

1) $A - BL$ stabil

2) $0.5 \leq A_m < \infty$ Fallst. x tillg

$\varphi_m \geq 60°$



3) Med Kalman filter

"None"

4) Cheap control

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

Introduction

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

T-AC Aug -78

SISO fallet

$$y(k) = \frac{B}{A} u$$

$$\Sigma (y^2 + \rho u^2)$$

Slutna systemet

$$H_c = C\left(zI - (\Phi - \Gamma L)\right)^{-1}\Gamma = \frac{B}{P}$$

där

$$\rho A(z^{-1})A(z) + B(z^{-1})B(z) = r P(z^{-1})P(z)$$

$$r = \Gamma^T S \Gamma + \rho$$