12

Optimal Design Methods: A Polynomial Approach

12.1 Introduction

Optimal design methods based on input-output models are considered in this chapter. Design of regulators based on linear models and quadratic criteria is discussed. This is one class of problems that admits closed-form solutions. The problems are solved by other methods in Chapter 11. The input-output approach gives additional insight and different numerical algorithms are also obtained.

The problem formulation is given in Sec. 12.2. This includes discussion of models for dynamics, disturbances, and criteria, as well as specification of admissible controls. The model is given in terms of three polynomials. A very simple example is also solved using first principles. This example shows clearly that optimal control and optimal filtering problems are closely connected. The prediction problem is then solved in Sec. 12.3. The solution is easily obtained by polynomial division. A simple explicit formula for the transfer function of the optimal predictor is given.

The minimum-variance control law is derived in Sec. 12.4. For systems with stable inverses, the control law is obtained in terms of the polynomials that characterize the optimal predictor. For systems with unstable inverses, the solution is obtained by solving a Diophantine equation in polynomials of the type discussed in Chapter 5. The minimum-variance control problem may thus be interpreted as a pole-placement problem. This gives insight into suitable choices of closed-loop poles and observer poles for the pole-placement problem. The LQG-control problem is solved in Sec. 12.5. It is shown that the solution may be expressed in terms of spectral factorization and solution of a Diophantine equation. Practical aspects, such as selection of the sampling period, are given in Sec. 12.6.

12.2 Problem Formulation

It is assumed that the process to be controlled is linear and time-invariant and that it has one input u and one output y. The dynamics of the process are characterized by a combination of a time-delay and a rational-transfer function. It is also assumed that the disturbances may be described as filtered white noise. A steady-state regulation problem is considered. The criterion is based on the mean-square deviations of the control signal and the output signal. In the formal problem statement given next, it is assumed that the model and the criterion are sampled [compare with Sec. 2.3 and (11.1)].

Process Dynamics

Assume that the process dynamics are characterized by

$$x(k) = \frac{B_1(q)}{A_1(q)}\, u(k) \tag{12.1}$$

where $A_1(q)$ and $B_1(q)$ are polynomials in the forward-shift operator.

Disturbances

Assume that the influence of the environment on the process can be characterized by disturbances that are stochastic processes. Because the system is linear, the principle of superposition can be used to reduce all disturbances to an equivalent disturbance v at the system output. The output of the system is thus given by

$$y(k) = x(k) + v(k) \tag{12.2}$$

Further assume that the disturbance v may be represented as the output of a linear system driven by white noise—that is,

$$v(k) = \frac{C_1(q)}{A_2(q)}\, e(k) \tag{12.3}$$

where $C_1(q)$ and $A_2(q)$ are polynomials in the forward-shift operator, and $e(k)$ is a sequence of independent or uncorrelated random variables with zero mean and standard deviation σ. The disturbance v may be a stationary random process. It may, however, also be drifting, because the polynomial $A_2(q)$ may be unstable. The model of the process and its environment can be reduced to a standard form. Eliminate v and x among (12.1), (12.2), and (12.3), and introduce

$$\begin{aligned} A &= A_1A_2 \\ B &= B_1A_2 \\ C &= C_1A_1 \end{aligned} \tag{12.4}$$

The following model is then obtained.

$$A(q)y(k) = B(q)u(k) + C(q)e(k) \tag{12.5}$$

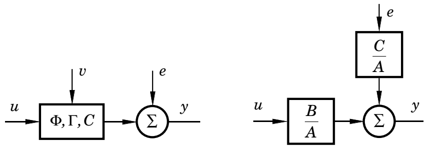

Figure 12.1 Representation of a system with one input and stochastic disturbances using one or two noise sources.

This is the canonical model, which will be the basis of the control design. In the special case when there are no disturbances, the model is simply a rational pulse-transfer function (see Sec. 2.6). When there is no control signal, the model is a stochastic process with a rational spectral density or an ARMA process (see Sec. 10.4). The model (12.5) is a convenient canonical representation of a linear system perturbed by noise. In Chapter 11 the process was driven by two noise sources. By using the spectral-factorization theorem (Theorem 10.3) the noise can be reduced to one source. Compare Fig. 12.1.

When the polynomial $C(q)$ has all its zeros inside the unit disc, it is called an *innovation's representation*, because the random variables $e(k)$ represent the innovations of the random process. Notice the symmetry between y and e. If e and u are known up to time k, then $y(k)$ can be computed, and if y and u are known up to time k, the innovation $e(k)$ can also be computed. Notice that the calculations of the residuals are governed by the dynamics of the polynomial $C(q)$. This polynomial can therefore be interpreted as the observer polynomial. Because (12.5) is an innovations model, the solutions to filtering problems become very simple.

Equation (12.5) can be normalized so that the leading coefficients of the polynomials $A(q)$ and $C(q)$ are unity. Such polynomials are called *monic*. The polynomial C may also be multiplied by an arbitrary power of q, as this does not change the correlation structure of $C(q)e(t)$. This may be used to normalize C so that $\deg C = \deg A$. The polynomials $A(q)$ and $B(q)$ may have zeros inside or outside the unit disc. It is assumed that all the zeros of the polynomial $C(q)$ are inside the unit disc. By spectral factorization (Theorem 10.3), the polynomial $C(q)$ may be changed so that all its zeros are inside the unit disc or on the unit circle. An example is used to show this important point.

Example 12.1 Modification of the polynomial C

Consider the polynomial

$$C(z) = z + 2$$

which has the zero $z = -2$ outside the unit disc. Consider the signal

$$n(k) = C(q)e(k)$$

where $e(k)$ is a sequence of uncorrelated random variables with zero mean and

unit variance. The spectral density of n is given by

$$\phi(e^{i\omega h}) = \frac{1}{2\pi} C(e^{i\omega h})C(e^{-i\omega h})$$

Because

$$\begin{aligned} C(z)C(z^{-1}) &= (z+2)(z^{-1}+2) = (1+2z^{-1})(1+2z) \\ &= (2z+1)(2z^{-1}+1) = 4(z+0.5)(z^{-1}+0.5) \end{aligned}$$

the signal n may also be represented as

$$n(k) = C^*(q)e(k)$$

where

$$C^*(z) = 2z + 1$$

is the reciprocal of the polynomial $C(z)$ (see Sec. 2.6). ■

If the calculations of (12.4) give a polynomial $C(q)$ that has zeros outside the unit disc, the polynomial C is factored as

$$C = C^+C^-$$

where C^- contains all factors with zeros outside the unit disc. The polynomial C is then replaced by C^+C^{-*}.

Criteria

In steady-state regulation it makes sense to express the criteria in terms of *steady-state variances* of the control variable and the process output. For regulation of systems with one output, the criterion may be to minimize the variance of the output. This is discussed in Sec.6.6 . Also compare with Fig. 6.7. This leads to the criterion

$$J_{mv} = \mathrm{E}y^2(k) \tag{12.6}$$

where it is assumed that the scales are chosen so that $y = 0$ corresponds to the desired set point. A control law that minimizes the criterion (12.6) is called *minimum-variance control*. The criterion may also be expressed as

$$J_\infty = \lim_{N\to\infty} \mathrm{E}\left\{\frac{1}{N}\sum_{k=1}^{N} y^2(k)\right\}$$

Notice that this criterion is an approximation of the continuous-time loss function

$$J_\infty = \lim_{T\to\infty} \mathrm{E}\left\{\frac{1}{T}\int_0^T y^2(t)\,dt\right\} \tag{12.7}$$

A more accurate approximation, which takes the behavior of the signals between the sampling instants into account, is given in Sec. 11.1. Some consequences of the approximation are discussed in Sec. 12.6. The properties of the control signal under minimum-variance control depend critically on the sampling period. A short sampling period gives a large variance of the control signal and a long sampling period gives a small variance.

In some cases it is desired to trade variances of control and output signals. This may be done by introducing the loss function

$$J_{lq} = \mathrm{E}\Big(y^2(k) + \rho u^2(k)\Big) \tag{12.8}$$

The control law that minimizes this criterion is called the *linear quadratic control law*.

Admissible Controls

It is assumed that the control law is such that $u(k)$, that is, the value of the control signal at time k, is a function of $y(k), y(k-1), \dots$ and $u(k-1), u(k-2), \dots$. Thus the computational delay is negligible in comparison with the sampling period. It is very easy to modify the results to take delays in the computations into account.

There are two versions of the theory. A linear control law may be postulated. It is then sufficient to assume that the disturbances $e(i)$ and $e(j)$ are uncorrelated for $i \neq j$. If $e(i)$ and $e(j)$ are assumed to be independent, it can be shown that the optimal-control law is linear. The formula for the optimal-control law is the same in both cases.

Minimum-Variance Control: An Example

The optimal-control problem defined by the model of (12.5) and the criterion of (12.6) is solved in a special case. The solution, which is easily obtained from first principles, gives good insight into the assumptions made. It also indicates how the general problem should be solved.

Consider the first-order system

$$y(k+1) + a\,y(k) = bu(k) + e(k+1) + ce(k) \tag{12.9}$$

where $|c| < 1$ and $e(k)$ is a sequence of independent random variables with unit variance.

Consider the situation at time k. The outputs $y(k)$, $y(k-1), \dots$ have been observed. The control $u(k)$ should be determined so that the output is as close to zero as possible. It follows from (12.9) that $y(k+1)$ may be changed arbitrarily by a proper choice of $u(k)$. Because $e(k+1)$ is independent of $y(k)$ and of the terms of the right-hand side of (12.9), it follows that

$$\operatorname{var} y(k+1) \geq \operatorname{var} e(k+1) = 1 \tag{12.10}$$

The term $e(k)$ may be computed in terms of the known data $y(k)$, $y(k-1), \dots$ and $u(k-1)$, $u(k-2), \dots$. When the variables $y(k)$ and $e(k)$ are known, the control law

$$u(k) = \big(a\,y(k) - ce(k)\big)/b \tag{12.11}$$

gives

$$y(k+1) = e(k+1) \tag{12.12}$$

which corresponds to the lower bound in (12.10). If the control law in (12.11) is used in each step, Eq. (12.12) holds for all k. The computation of $e(k)$ from the data available at time k is then trivial and the control law in (12.11) can be written as

$$u(k) = -\frac{c-a}{b}\,y(k) \tag{12.13}$$

The optimal control is thus a proportional feedback with the gain $(c-a)/b$.

To analyze the properties of the closed-loop system under optimal control, eliminate u between (12.9) and (12.13). This gives

$$y(k+1) + c\,y(k) = e(k+1) + ce(k)$$

Notice that the closed-loop system has the characteristic polynomial

$$C(z) = z + c$$

This shows the importance of the assumption that the polynomial $C(z)$ is stable. This difference equation has the solution

$$y(k) = e(k) + (-c)^{k-k_0}\Big(y(k_0) - e(k_0)\Big)$$

Because c is less than one in magnitude, the last term goes to zero as $k - k_0$ increases toward infinity. Thus control law in (12.13) gives the minimum-variance in steady state.

With this result, some observations are possible. The quantity $-a\,y(k) + bu(k) + ce(k)$ can be interpreted as the best estimate of $y(k+1)$, given the data available at time k. The quantity $e(k+1)$ is the prediction error. The control law in (12.13) implies that the control signal is chosen so that the predicted value is equal to the reference value, which is zero in this case. The control error is then equal to the prediction error. The solution to the minimum-variance control problem is thus closely related to the solution of a prediction problem. Therefore, the prediction problem is solved before the solution of the general minimum-variance control problem is attempted.

12.3 Optimal Prediction

Prediction theory can be stated in many different ways, which differ in the assumptions made on the process, the criterion, and the admissible predictors. One formulation is given in Sec. 11.3. In this section the following assumptions are made:

- The process to be predicted is generated by filtered white Gaussian noise.

- The best predictor is the one that minimizes the mean-square prediction error.
- An admissible m-step predictor for $y(k+m)$ is an arbitrary function of $y(k)$, $y(k-1), \ldots$.

An intuitive derivation of a predictor is first given. The result is then formalized.

Heuristics

Consider the signal y generated by the model

$$y(k) = \frac{C(q)}{A(q)}\, e(k) = \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k) \tag{12.14}$$

where A^* and C^* are the reciprocals of A and C, that is, $A^*(q^{-1}) = q^{-n}A(q)$, and q^{-1} is the backward-shift operator. It is convenient to introduce this operator because the discussion is based on causality. It is assumed that A and C are of order n.

Consider the situation at time k. The variables $y(k)$, $y(k-1), \ldots$ have been observed and it is desired to predict $y(k+m)$. A formal series expansion of C^*/A^* in q^{-1} gives

$$\begin{aligned} y(k+m) &= \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k+m) \\ &= \underbrace{e(k+m) + f_1 e(k+m-1) + \cdots + f_{m-1} e(k+1)}_{\text{Unknown at time } k} \\ &\quad + \underbrace{f_m e(k) + f_{m+1} e(k-1) + \cdots}_{\text{Known at time } k} \end{aligned} \tag{12.15}$$

The terms of the right-hand side are all independent because $e(k)$ is a sequence of independent random variables. It follows from the model of (12.14) that if the polynomial C is stable, then $e(i)$ can be computed exactly from $y(i), y(i-1), \ldots$ using

$$e(k) = \frac{A^*(q^{-1})}{C^*(q^{-1})}\, y(k)$$

The first terms of (12.15) are independent of the data at time k. The second part is known functions of the data available at time k. Thus it follows that the optimal predictor is given by

$$\hat{y}(k+m \mid k) = f_m e(k) + f_{m+1} e(k-1) + f_{m+2} e(k-2) + \cdots$$

and that the prediction error is

$$\tilde{y}(k+m \mid k) = e(k+m) + f_1 e(k+m-1) + \cdots + f_{m-1} e(k+1)$$

To provide a formal proof it remains to show how the numbers f_i can be computed from A and C and how $e(k)$ can be expressed in terms of past data.

Main Result

The main result can be stated as follows.

THEOREM 12.1 OPTIMAL PREDICTION Let $y(k)$ be a random process generated by the model in (12.14), where all the zeros of the polynomial $C(z)$ are inside the unit disc, and $e(k)$ is a sequence of independent random variables. The minimum-variance predictor over m steps is given by

$$\hat{y} = \hat{y}(k+m \mid k) = \frac{qG(q)}{C(q)}\, y(k) = \frac{G^*(q^{-1})}{C^*(q^{-1})}\, y(k) \tag{12.16}$$

where the polynomials F and G are the quotient and the remainder when dividing $q^{m-1}C$ by A; that is,

$$q^{m-1}C(q) = A(q)F(q) + G(q) \tag{12.17}$$

The prediction error is a moving average

$$\tilde{y}(k+m \mid k) = y(k+m) - \hat{y}(k+m \mid k) = F(q)e(k+1) \tag{12.18}$$

It has zero mean and the variance

$$\mathrm{E}\tilde{y}(k+m \mid k)^2 = \left(1 + f_1^2 + \cdots + f_{m-1}^2\right)\sigma^2 \tag{12.19}$$

Proof. The polynomial F is monic of degree $m-1$ and G is of degree less than n. Hence

$$\begin{aligned} F(q) &= q^{m-1} + f_1 q^{m-2} + \cdots + f_{m-1} \\ G(q) &= g_0 q^{n-1} + g_1 q^{n-2} + \cdots + g_{n-1} \end{aligned}$$

We introduce

$$\begin{aligned} F^*(q^{-1}) &= 1 + f_1 q^{-1} + \cdots + f_{m-1} q^{-m+1} \\ G^*(q^{-1}) &= g_0 + g_1 q^{-1} + \cdots + g_{n-1} q^{-n+1} \end{aligned}$$

It follows from (12.17) that

$$C^*(q^{-1}) = A^*(q^{-1})F^*(q^{-1}) + q^{-m}G^*(q^{-1}) \tag{12.20}$$

Equation (12.15) can then be written as

$$y(k+m) = \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k+m) = F^*(q^{-1})e(k+m) + \frac{G^*(q^{-1})}{A^*(q^{-1})}\, e(k)$$

By using Equation (12.14) the signal e in the last term can be expressed in terms of the data available at time k. Hence,

$$y(k+m) = F^*(q^{-1})e(k+m) + \frac{G^*(q^{-1})}{C^*(q^{-1})}\, y(k)$$

The first term of the right-hand side is a linear function of $e(k+1)$, $e(k+2)$, ..., $e(k+m)$, which are all independent of the data $y(k)$, $y(k-1)$, $y(k-2)$, ... available at time k. The last term is a linear function of the data. Let $\hat{y}$ be an arbitrary function of $y(k)$, $y(k-1)$, ... Then

$$\begin{aligned}\mathrm{E}\Big(y(k+m)-\hat{y}\Big)^2 &= \mathrm{E}\Big(F^*(q^{-1})e(k+m)\Big)^2 + \mathrm{E}\left(\frac{G^*(q^{-1})}{C^*(q^{-1})}\,y(k)-\hat{y}\right)^2 \\ &\quad + 2\mathrm{E}\left\{\Big(F^*(q^{-1})e(k+m)\Big)\left(\frac{G^*(q^{-1})}{C^*(q^{-1})}\,y(k)-\hat{y}\right)\right\}\end{aligned} \tag{12.21}$$

The last term is zero because $e(k+m), e(k+m-1), \ldots,$ and $e(k+1)$ have zero mean values and are independent of $y(k), y(k-1), \ldots$. The predictor that minimizes the mean-square prediction error is thus given by (12.16) and the prediction error by (12.18). The proof is completed by taking the mean value of the square of the prediction error (12.18). This gives (12.19). ■

Remark 1. Notice that the best predictor is linear. The linearity does not depend critically on the minimum-variance criterion. If the probability density of $y(k)$ is symmetric, the predictor of (12.16) is optimal for all criteria of the form $\mathrm{E}\,g\big((y(k+m)-\hat{y})^2\big)$ for symmetric g.

Remark 2. The assumption that $e(i)$ and $e(j)$ are independent for $i \neq j$ is essential for the last term in (12.21) to vanish. If the variables are uncorrelated, the term will still vanish if the predictor $\hat{y}$ is restricted to being linear.

Remark 3. It follows from (12.18) that

$$\tilde{y}(k+1 \mid k) = y(k+1) - \hat{y}(k+1 \mid k) = e(k+1)$$

The random variables $e(k)$ can thus be interpreted as the innovations of the process $y(k)$ (compare with Sec. 10.4).

Remark 4. Notice that the function

$$J(m) = \sigma^2\left(1 + f_1^2 + \cdots + f_{m-1}^2\right)$$

is the variance of the prediction error over the time interval mh. The function $J(m)$ approaches the variance of y as $m \to \infty$. A graph of the function J shows how well the process may be predicted over different horizons. See Example 12.3.

Remark 5. The predictor discussed in this section is equivalent to the steady-state predictor obtained using the Kalman filter in Sec.11.3 (see Example 11.6).

Calculation of the Optimal Predictor

It follows from (12.17) that $F(q)$ is the quotient and $G(q)$ the remainder when dividing $q^{m-1}C(q)$ by $A(q)$. The polynomials F and G can thus be determined by

polynomial division. An explicit formula for the coefficients of the polynomials can also be given. Equating the coefficients of equal powers of q in (12.17) gives the following equations:

$$\begin{aligned}
c_1 &= a_1 + f_1 \\
c_2 &= a_2 + a_1 f_1 + f_2 \\
&\ \vdots \\
c_{m-1} &= a_{m-1} + a_{m-2} f_1 + \cdots + a_1 f_{m-2} + f_{m-1} \\
c_m &= a_m + a_{m-1} f_1 + \cdots + a_1 f_{m-1} + g_0 \\
c_{m+1} &= a_{m+1} + a_m f_1 + \cdots + a_2 f_{m-1} + g_1 \\
&\ \vdots \\
c_n &= a_n + a_{n-1} f_1 + \cdots + a_{n-m+1} f_{m-1} + g_{n-m} \\
0 &= a_n f_1 + a_{n-1} f_2 + \cdots + a_{n-m+2} f_{m-1} + g_{n-m+1} \\
&\ \vdots \\
0 &= a_n f_{m-1} + g_{n-1}
\end{aligned}$$

These equations are easy to solve recursively. Compare the solution of the Diophantine equation in Chapter 5.

Example 12.2 Prediction

Consider the system (12.14) defined by the polynomials

$$\begin{aligned} A(q) &= q^2 - 1.5q + 0.7 \\ C(q) &= q^2 - 0.2q + 0.5 \end{aligned}$$

and where e has unit variance. Determine first the three-step-ahead prediction of the output. The identity (12.17) gives

$$q^2(q^2 - 0.2q + 0.5) = (q^2 - 1.5q + 0.7)(q^2 + f_1 q + f_2) + g_0 q + g_1$$

This gives the triangular linear system of equations

$$\begin{aligned}
q^3:&\quad -0.2 = -1.5 + f_1 && f_1 = 1.3 \\
q^2:&\quad 0.5 = 0.7 - 1.5 f_1 + f_2 && f_2 = 1.75 \\
q^1:&\quad 0 = 0.7 f_1 - 1.5 f_2 + g_0 && g_0 = 1.715 \\
q^0:&\quad 0 = 0.7 f_2 + g_1 && g_1 = -1.225
\end{aligned}$$

The prediction three steps ahead is thus given by

$$\hat{y}(k+3 \mid k) = \frac{qG(q)}{C(q)}\,y(k) = \frac{1.715q^2 - 1.225q}{q^2 - 0.2q + 0.5}\,y(k)$$

and the variance of the prediction error is

$$\mathrm{E}\tilde{y}^2 = 1 + (1.3)^2 + (1.75)^2 = 5.7525$$

■

Example 12.3 Influence of prediction horizon

Consider the process in Example 12.2. From (12.19) it follows that the variance of the prediction error will increase with the prediction horizon. Also (12.17) shows that the F-polynomial is obtained from the division of the C- and A-polynomials. That is, the coefficients f_i are the coefficients of the impulse response of the system. Thus

$$\begin{aligned} y(k) &= \frac{C(q)}{A(q)}\,e(k) = \frac{q^2 - 0.2q + 0.5}{q^2 - 1.5q + 0.7}\,e(k) \\ &= \left(1 + 1.3q^{-1} + 1.75q^{-2} + 1.715q^{-3} + \cdots\right) e(k) = \sum_{j=0}^{\infty} f_j e(k-j) \end{aligned}$$

and the prediction loss is

$$\mathrm{E}\tilde{y}^2(k+m \mid k) = \sigma^2 \sum_{j=0}^{m-1} f_j^2$$

Figure 12.2 shows the variance of the prediction error for different values of the prediction horizon m. It is seen that the variance of the prediction error is monotonically increasing with m. Figure 12.3 shows the output, the predicted output, and the accumulated prediction loss, $\sum\left(y(k) - \hat{y}(k \mid k-m)\right)^2$, for different prediction horizons. ■

The Case When C Has Zeros on the Unit Circle

The predictor of (12.16) is a dynamic system with the characteristic polynomial $C(z)$. The assumption that C has all its zeros inside the unit disc thus guarantees that the predictor is stable in steady state. The initial conditions are irrelevant because their influence will decay exponentially.

It follows from the spectral factorization that C may be chosen to have its zeros inside the unit disc or on the unit circle. The zeros outside the unit disc is mirrored in the unit circle. Compare with Example 12.1. Thus it remains to discuss the case when C has zeros on the unit circle.

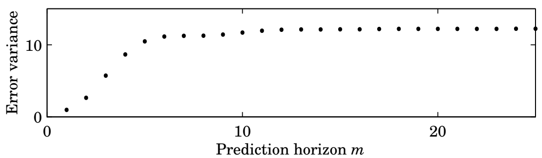

Figure 12.2 The variance of the prediction error as function of the prediction horizon m for the system in Example 12.3.

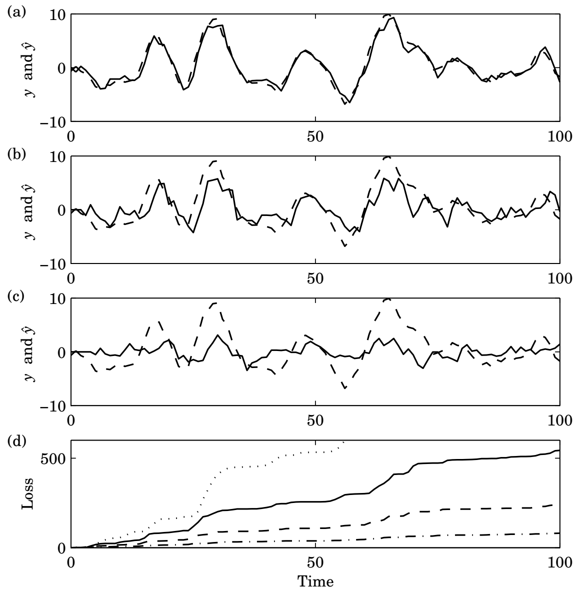

Figure 12.3 The process output (dashed) and the predicted output (solid) for Example 12.3 when (a) $m = 1$, (b) $m = 3$, (c) $m = 5$, and (d) the accumulated prediction loss, $\sum\left(y(k) - \hat{y}(k \mid k-m)\right)^2$, for $m = 1$ (dashed-dotted), $m = 2$ (dashed), $m = 3$ (solid), and $m = 5$ (dotted).

Example 12.4 Zeros on the unit circle

Consider the process

$$y(k) = e(k) - e(k-1) \tag{12.22}$$

In this case the polynomial $C(z) = z - 1$ has a zero on the unit circle. Applying the previous methods formally gives the one-step predictor

$$\hat{y}(k+1 \mid k) = -e(k)$$

Attempting to calculate $e(k)$ from $y(k), y(k-1), \ldots, y(k_0)$ as was done previously gives

$$e(k) = e(k_0 - 1) + \sum_{i=k_0}^{k} y(i) = e(k_0 - 1) + z(k)$$

The presence of the term $e(k_0 - 1)$, which does not go to zero as $k_0 \to -\infty$, shows the consequences of C being unstable. The Kalman filtering theory can, however, be used to determine the optimal predictor. The signal given by (12.22) can be written as

$$\begin{aligned} x(k+1) &= e(k) \\ y(k) &= -x(k) + e(k) \end{aligned}$$

where $R_1 = R_2 = R_{12} = \sigma^2$ with the notations used in Sec. 11.3. The Kalman filter is

$$\begin{aligned} \hat{x}(k+1 \mid k) &= K(k)\Big(y(k) + \hat{x}(k \mid k-1)\Big) \\ P(k+1) &= \frac{\sigma^2 P(k)}{P(k) + \sigma^2} \\ K(k) &= \frac{\sigma^2}{P(k) + \sigma^2} \end{aligned}$$

with the initial conditions

$$\begin{aligned} \hat{x}(k_0 \mid k_0 - 1) &= 0 \\ P(k_0) &= \sigma^2 \end{aligned}$$

The predictor for the output is

$$\hat{y}(k+1 \mid k) = -\hat{x}(k+1 \mid k) = -K(k)\Big(y(k) - \hat{y}(k \mid k-1)\Big)$$

Simple calculations give

$$\hat{y}(k+1 \mid k) = -\frac{1}{k - k_0 + 2} \sum_{n=0}^{k-k_0} (n+1) y(k_0 + n)$$

The optimum predictor is thus a time-varying system. Notice that the influence of the initial condition $y(k_0)$ goes to zero at the rate $1/(k - k_0 + 2)$. This is much slower than in the case of stable polynomials C. ■

It follows from the example that the optimal predictor is a time-varying system if the polynomial C has zeros on the unit circle. Such models should be avoided if time-invariant predictors are desired. Unfortunately, this fact is not always noticed, as Example 12.5 illustrates.

Example 12.5 How to model offsets

The model

$$A(q)y(k) = C(q)e(k) + b$$

where b is an unknown constant, represents a signal with an offset. The constant b can be eliminated by taking differences. Hence,

$$(q-1)A(q)y(k) = (q-1)C(q)e(k)$$

The common factor $q - 1$ can be eliminated by regarding $\Delta y(k) = (q-1)y(k)$ as the output. The model

$$A(q)\Delta y(k) = (q-1)C(q)e(k) = \tilde{C}(q)e(k)$$

is then obtained. In this model the polynomial $\tilde{C}$ apparently has a zero on the unit circle. This model is, however, not very desirable because the optimal predictor is a time-varying system. It is much better to model an offset as a Wiener process. This leads to a process model with $A(1) = 0$ that is unstable with a stationary predictor. ■

Other reasons for avoiding models where the polynomial $C(z)$ has zeros close to the unit circle are given in Sec. 12.6.

12.4 Minimum-Variance Control

To determine the minimum-variance control law, the special case when the polynomial B in (12.5) is stable is discussed first. This means that the process dynamics have a stable inverse. With some abuse of language, this case is also called the minimum-phase case because the pulse-transfer function has all its zeros inside the unit disc. The solution to the control problem is very simple in this special case. The solution also gives insight into the properties of the control problem.

Systems with Stable Inverses

By introducing the backward-shift operator q^{-1}, the model in (12.5) can be written as

$$\begin{aligned} y(k) &= \frac{B(q)}{A(q)}\, u(k) + \frac{C(q)}{A(q)}\, e(k) \\ &= \frac{B^*(q^{-1})}{A^*(q^{-1})}\, q^{-d} u(k) + \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k) \end{aligned} \tag{12.23}$$

where

$$d = \deg A - \deg B > 0$$

is the *pole excess* of the system (see Sec. 2.6). Further, $\deg A = \deg C = n$. The reciprocal polynomials are introduced to make the discussion based on causality arguments more transparent.

It follows from (12.23) that

$$\begin{aligned} y(k+d) &= \frac{C^*(q^{-1})}{A^*(q^{-1})}\, e(k+d) + \frac{B^*(q^{-1})}{A^*(q^{-1})}\, u(k) \\ &= F^*(q^{-1})e(k+d) + \frac{G^*(q^{-1})}{A^*(q^{-1})}\, e(k) + \frac{B^*(q^{-1})}{A^*(q^{-1})}\, u(k) \end{aligned} \tag{12.24}$$

where Equation (12.20) with $m = d$ has been used to obtain the last equality. The first term of the right-hand side is independent of the data available at time k and thus also of the second and third terms. The second term can be computed exactly in terms of data available at time k. To do this, the variable $e(k)$ is given by (12.23); that is,

$$e(k) = \frac{A^*}{C^*}\, y(k) - q^{-d}\,\frac{B^*}{C^*}\, u(k)$$

where the arguments of the polynomials have been dropped to simplify the writing. Using this expression for e, Eq. (12.24) can be written as

$$\begin{aligned} y(k+d) &= F^* e(k+d) + \frac{G^*}{C^*}\, y(k) - q^{-d}\,\frac{B^*G^*}{A^*C^*}\, u(k) + \frac{B^*}{A^*}\, u(k) \\ &= F^* e(k+d) + \frac{G^*}{C^*}\, y(k) + \frac{B^*F^*}{C^*}\, u(k) \end{aligned} \tag{12.25}$$

Now let $u(k)$ be an arbitrary function of $y(k), y(k-1), \ldots$ and $u(k-1), u(k-2), \ldots$. Then

$$\mathrm{E}y^2(k+d) = \mathrm{E}\big(F^* e(k+d)\big)^2 + \mathrm{E}\left(\frac{G^*}{C^*}\, y(k) + \frac{B^*F^*}{C^*}\, u(k)\right)^2 \tag{12.26}$$

The mixed terms vanish because $e(k+d), \ldots, e(k+1)$ are independent of $y(k)$, $y(k-1), \ldots$ and $u(k), u(k-1), \ldots$. Because the last term in (12.26) is nonnegative, it follows that

$$\mathrm{E}y^2(k+d) \geq (1 + f_1^2 + \cdots + f_{d-1}^2)\sigma^2$$

where equality is obtained for

$$u(k) = -\frac{G^*(q^{-1})}{B^*(q^{-1})F^*(q^{-1})}\, y(k) = -\frac{G(q)}{B(q)F(q)}\, y(k) \tag{12.27}$$

which is the desired minimum-variance control law. The result can be summarized as follows.

THEOREM 12.2 MINIMUM-VARIANCE CONTROL—STABLE INVERSE Consider a process described by (12.5), where $e(k)$ is a sequence of independent random variables with zero mean values and standard deviations σ. Let the polynomials B and C have all their zeros inside the unit disc. The minimum-variance control law is then given by (12.27), where the polynomials F^* and G^* are given by (12.20) with $m = d$. This control law gives the output

$$y(k) = F^*(q^{-1})e(k) = e(k) + f_1 e(k-1) + \cdots + f_{d-1} e(k-d+1)$$

in steady state. ■

Remark 1. The theorem still holds when $e(i)$ and $e(j)$ are uncorrelated for $i \neq j$ if a linear control law is postulated.

Remark 2. The result is closely related to the solution of the prediction problem (Theorem 12.1). Identity (12.17) or (12.20) was used in both cases. The last two terms in (12.25) can be interpreted as the d-step prediction of the output. The minimum-variance strategy is thus obtained by predicting the output d steps ahead and choosing a control that makes the prediction equal to the desired output. The stochastic-control problem can thus be separated into two problems, one stochastic-prediction problem and one deterministic-control problem. Theorem 12.2 can therefore be interpreted as a separation theorem.

Remark 3. The error under minimum-variance control is a moving average of order $d-1$. Thus the covariance function of the regulation error will vanish for arguments larger than $d-1$. This fact can be used for diagnosis to determine if a minimum-variance strategy is used.

Remark 4. All process zeros are canceled when the control law of (12.27) is used. The consequences of this are discussed later.

It is very easy to calculate the minimum-variance control law for a given model (12.5), as illustrated by the following example.

Example 12.6 Minimum-variance control

Consider a system given by (12.5), where

$$\begin{aligned} A(q) &= q^3 - 1.7q^2 + 0.7q \\ B(q) &= q + 0.5 \\ C(q) &= q^3 - 0.9q^2 \end{aligned}$$

The pole excess is $d = 2$. Division of $q^{d-1}C(q)$ by $A(q)$ gives the quotient

$$F(q) = q + 0.8$$

and the remainder

$$G(q) = 0.66q^2 - 0.56q$$

The minimum-variance control law is thus

$$u(k) = -\frac{q(0.66q - 0.56)}{(q+0.5)(q+0.8)}\, y(k)$$

The variance of the output when the optimal controller is used is

$$\mathrm{E}y^2 = 1 + (0.8)^2 = 1.64$$

■

Example 12.7 Influence of the delay

Let the process be described by

$$A^*(q^{-1}) = 1 - 1.5q^{-1} + 0.7q^{-2}$$
$$B^*(q^{-1}) = q^{-d}(1 + 0.5q^{-1})$$
$$C^*(q^{-1}) = 1 - 0.2q^{-1} + 0.5q^{-2}$$

Compute the minimum-variance controller when $d = 1$, 3, or 5. The controller is given by (12.27), where the F-polynomial is given in Example 12.3. Figure 12.4 shows the output and input when the minimum-variance controller is used for different delays in the process. When $d = 1$, $d = 3$, and $d = 5$ the output variance is 1, 5.8, and 10.5, respectively. ■

Interpretation as Pole-Placement Design

The minimum-variance control law can be interpreted in terms of the pole-placement design discussed in Chapter 5. To see the relationships, the closed-loop system obtained when the control law of (12.27) is applied to the system

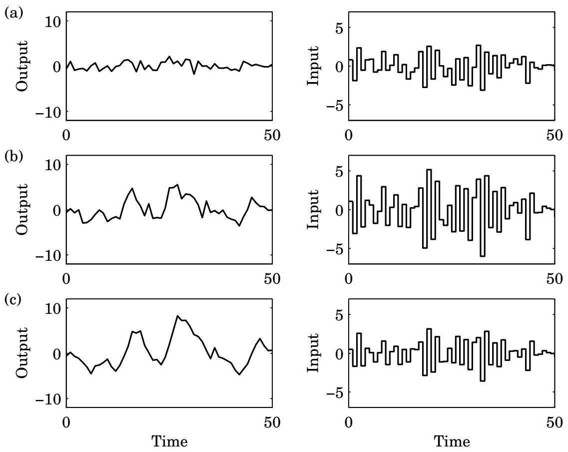

Figure 12.4 Simulation of the system in Example 12.7 with the control law given by Theorem 12.2. The output (left) and the input (right) when (a) $d = 1$, (b) $d = 3$, and (c) $d = 5$.

of (12.5) is analyzed. Equations (12.5) and (12.27) can be written as

$$\begin{pmatrix} A(q) & -B(q) \\ G(q) & F(q)B(q) \end{pmatrix} \begin{pmatrix} y(k) \\ u(k) \end{pmatrix} = \begin{pmatrix} C(q) \\ 0 \end{pmatrix} e(k) \qquad (12.28)$$

The characteristic polynomial of the closed-loop system is the determinant of the matrix on the left-hand side of (12.28). Hence,

$$A(q)F(q)B(q) + G(q)B(q) = q^{d-1}B(q)C(q) \qquad (12.29)$$

where Eq. (12.17), with $m = d$, is used to obtain the first equality. The closed-loop system is of order $2n - 1$. It has $2n - d$ poles at the zeros of B and C and an additional $d - 1$ poles at the origin.

The minimum-variance control strategy can be interpreted as a pole-placement design, where the poles are placed at the zeros given by (12.29). The similarities to pole placement are seen even more clearly if the control law of (12.27) is written as

$$u(k) = -\frac{G(q)}{B(q)F(q)}\, y(k) = -\frac{S(q)}{R(q)}\, y(k)$$

where $S = G$ and $R = FB$ [compare with Eq. (5.2)]. Multiplication of (12.17) by B gives

$$q^{d-1}C(q)B(q) = A(q)F(q)B(q) + G(q)B(q) = A(q)R(q) + B(q)S(q)$$

This equation is a special case of the Diophantine equation in (5.22) when $B^+ = B$ and with $A_c = q^{d-1}B$ and $A_o = C$.

Systems with Unstable Inverses

Remark 4 to Theorem 12.2 mentions that the control law given by (12.27) cancels all process zeros. If there are process zeros outside the unit disc, the closed-loop system will then have unstable modes that are unobservable from the output. The implications of this are discussed first. Other control laws that do not require all zeros of $B(z)$ to be inside the unit disc are then presented.

Solving Eq. (12.28) for y and u gives

$$y(k) = \frac{F(q)}{q^{d-1}}\, e(k)$$

and

$$u(k) = -\frac{G(q)}{q^{d-1}B(q)}\, e(k)$$

The necessity of the assumption that B is stable is clearly seen from these equations. If the polynomial B is unstable, the system has unstable modes, which are excited by the disturbance. These unstable modes are coupled to the control signal and the control signal grows exponentially. However, the output signal remains bounded because the unstable modes are not coupled to the output. An example illustrates what happens.

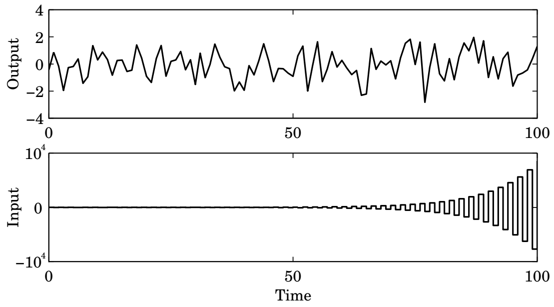

Figure 12.5 Simulation of the system in Example 12.8 with the control law given by Theorem 12.2 that cancels an unstable process zero.

Example 12.8 Cancellation of unstable process zero

Consider a system described by the polynomials

$$A(z) = (z-1)(z-0.7)$$
$$B(z) = 0.9z + 1$$
$$C(z) = z(z-0.7)$$

The polynomial $B(z)$ has a zero $z = -10/9$, which is outside the unit disc. A simulation when using (12.27) is shown in Fig. 12.5. The presence of the unstable mode is clearly seen in the control signal, although it is not noticeable in the system output. If the simulation is continued, the control signal will finally be so large that overflow or numerical errors occur. In a practical problem the signal will quickly be so large that the linear approximation is no longer valid. After a short time the unstable mode will then be noticeable in the output. ■

The minimum-variance control law is extended to the case when the polynomial B has zeros outside the unit disc in Theorem 12.3.

THEOREM 12.3 MINIMUM-VARIANCE CONTROL—GENERAL CASE Consider a system described by (12.5). Factor the polynomial $B(z)$ as

$$B(z) = B^+(z)B^-(z) \tag{12.30}$$

where $B^{-*}(z)$ is monic. All zeros of the polynomial $B^+(z)$ are inside the unit disc and all zeros of $B^-(z)$ are outside the unit disc or on the unit circle. Assume that all the zeros of polynomial $C(z)$ are inside the unit disc and that the polynomials $A(z)$ and $B^-(z)$ do not have any common factors. The minimum-variance control law is then given by

$$u(k) = -\frac{G(q)}{B^+(q)F(q)}\,y(k) \tag{12.31}$$

where $F(q)$ and $G(q)$ are polynomials that satisfy the Diophantine equation

$$q^{d-1}C(q)B^{-*}(q) = A(q)F(q) + B^-(q)G(q) \tag{12.32}$$

in which $\deg F = d + \deg B^- - 1$ and $\deg G < \deg A = n$.

Proof. The proof is based on a clever trick introduced by Wiener in his original work on prediction. An alternative method is used in the proof of Theorem 12.4. Consider the operator

$$\frac{1}{q+a}$$

where $|a| > 1$. This operator is normally interpreted as a causal unstable (unbounded) operator. Because $|a| > 1$ and the shift operator has the norm $\|q\| = 1$, the series expansion

$$\frac{1}{q+a} = \frac{1}{a}\,\frac{1}{1+q/a} = \frac{1}{a}\left(1 - \frac{q}{a} + \frac{q^2}{a^2} - \cdots\right)$$

converges. Thus the operator $(q+a)^{-1}$ can be interpreted as a noncausal stable operator; that is,

$$\frac{1}{q+a}\,y(k) = \frac{1}{a}\left(y(k) - \frac{1}{a}\,y(k+1) + \frac{1}{a^2}\,y(k+2) - \cdots\right)$$

With this interpretation, it follows that

$$(q+a)\left(\frac{1}{q+a}\,y(k)\right) = y(k)$$

The calculations required for the proof are conveniently done using the backward-shift operator. It follows from the process model of (12.5) that

$$y(k+d) = \frac{B^*(q^{-1})}{A^*(q^{-1})}\,u(k) + \frac{C^*(q^{-1})}{A^*(q^{-1})}\,e(k+d)$$

We introduce

$$w(k) = \frac{B^-(q^{-1})}{B^{-*}(q^{-1})}\,y(k)$$

where the operator $1/B^{-*}(q^{-1})$ is interpreted as a noncausal stable operator. The signals y and w have the same steady-state variance because B^- and B^{-*} are reciprocal polynomials and

$$\left|\frac{B^-(e^{-i\omega})}{B^{-*}(e^{-i\omega})}\right| = 1$$

An admissible control law that minimizes the variance of w also minimizes the variance of y. It follows that

$$w(k+d) = \frac{B^{+*}(q^{-1})B^{-}(q^{-1})}{A^*(q^{-1})} u(k) + \frac{C^*(q^{-1})B^{-}(q^{-1})}{A^*(q^{-1})B^{-*}(q^{-1})} e(k+d) \qquad (12.33)$$

The assumption that $A(z)$ and $B^-(z)$ are relatively prime guarantees that (12.32) has a solution. Equation (12.32) implies that

$$C^*(q^{-1})B^-(q^{-1}) = A^*(q^{-1})F^*(q^{-1}) + q^{-d}B^{-*}(q^{-1})G^*(q^{-1})$$

Division by A^*B^{-*} gives

$$\frac{C^*(q^{-1})B^-(q^{-1})}{A^*(q^{-1})B^{-*}(q^{-1})} = \frac{F^*(q^{-1})}{B^{-*}(q^{-1})} + q^{-d}\frac{G^*(q^{-1})}{A^*(q^{-1})}$$

By using this equation, (12.33) can be written as

$$w(k+d) = \frac{F^*(q^{-1})}{B^{-*}(q^{-1})} e(k+d) + \frac{B^{+*}(q^{-1})B^-(q^{-1})}{A^*(q^{-1})} u(k) + \frac{G^*(q^{-1})}{A^*(q^{-1})} e(k) \quad (12.34)$$

Because the operator $1/B^{-*}(q^{-1})$ is interpreted as a bounded noncausal operator and because $\deg F^* = d + \deg B^- - 1$, it follows that

$$\frac{F^*(q^{-1})}{B^{-*}(q^{-1})} e(k+d) = \alpha_1 e(k+1) + \alpha_2 e(k+2) + \cdots$$

These terms are all independent of the last two terms in (12.34). Using the arguments given in detail in the proof of Theorem 12.2, we find that the optimal control law is obtained by putting the sum of the last two terms in (12.34) equal to zero. This gives

$$u(k) = -\frac{G^*(q^{-1})}{B^{+*}(q^{-1})B^-(q^{-1})} e(k) \qquad (12.35)$$

and

$$y(k) = \frac{B^{-*}(q^{-1})}{B^-(q^{-1})} w(k) = \frac{F^*(q^{-1})}{B^-(q^{-1})} e(k) = \frac{F(q)}{q^{d-1}B^{-*}(q)} e(k) \qquad (12.36)$$

Elimination of $e(k)$ between (12.35) and (12.36) gives

$$u(k) = -\frac{G^*(q^{-1})}{B^{+*}(q^{-1})F^*(q^{-1})} y(k)$$

The numerator and the denominator have the same degree because $\deg G < n$ and the control law can then be rewritten as (12.31). ■

Remark 1. Only the stable process zeros are canceled by the optimal control law.

Remark 2. It follows from the proofs of Theorems 12.2 and 12.3 that the variance of the output of a system such as (12.5) may have several local minima if the polynomial $B(z)$ has zeros outside the unit disc. There is one absolute minimum given by Theorem 12.2. However, this minimum will give control signals that are infinitely large. The local minimum given by Theorem 12.3 is the largest of the local minima. The control signal is bounded in this case.

Remark 3. The factorization of (12.30) is arbitrary because B^+ could be multiplied by a number and B^- could be divided by the same number. It is convenient to select the factors so that the polynomial $B^{-*}(q)$ is monic.

Example 12.9 Minimum-variance control with unstable process zero

Consider the system in Example 12.8 where $d = 1$ and

$$\begin{aligned} B^+(z) &= 1 \\ B^-(z) &= B(z) \\ B^{-*}(z) &= z + 0.9 \end{aligned}$$

Equation (12.32) becomes

$$z(z-0.7)(z+0.9) = (z-1)(z-0.7)(z+f_1) + (0.9z+1)(g_0 z + g_1)$$

Let $z = 0.7$, $z = 1$, and $z = -10/9$. This gives

$$\begin{aligned} 0.7g_0 + g_1 &= 0 \\ g_0 + g_1 &= 0.3 \\ f_1 &= 1 \end{aligned}$$

The control law thus becomes

$$u(k) = -\frac{G(q)}{B^+(q)F(q)} y(k) = -\frac{q-0.7}{q+1} y(k)$$

The output is

$$y(k) = \frac{F(q)}{B^{-*}(q)} e(k-d+1) = \frac{q+1}{q+0.9} e(k) = e(k) + \frac{0.1}{q+0.9} e(k)$$

The variance of the output is

$$\mathrm{E}\, y^2 = \left(1 + \frac{0.1^2}{1-0.9^2}\right)\sigma^2 = \frac{20}{19}\sigma^2 = 1.05\sigma^2$$

which is about 5% larger than using the controller in Example 12.8. The variance of the control signal is $275\sigma^2/19 = 14.47\sigma^2$. A simulation of the control law is shown in Fig. 12.6. The figure that the controller performs well. Compare also with Fig. 12.5, which shows the effect of canceling the unstable zero. Figure 12.7 shows the accumulated output loss $\sum y^2(k)$ and input loss $\sum u^2(k)$ when the controllers in Example 12.8 and this example are used. The controller (12.27) gives lower output loss, but an exponentially growing input loss, and the controller based on (12.31) gives an accumulated input loss that grows linearly with time. ■

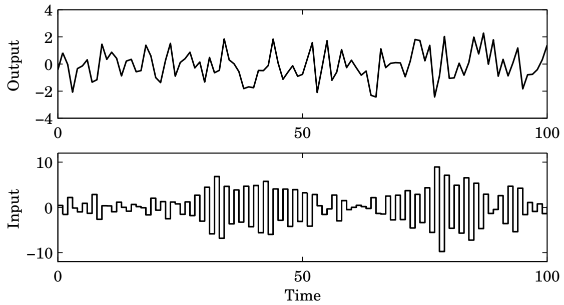

Figure 12.6 Simulation of the system in Example 12.9.

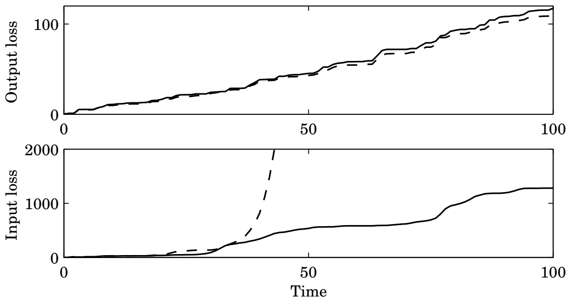

Figure 12.7 The accumulated output loss $\sum y^2(k)$ and input loss $\sum u^2(k)$ when the controllers (12.31) (solid) and (12.27) (dashed) are used.

A Pole-Placement Interpretation

Simple calculations show that the characteristic equation of the closed-loop system obtained from (12.5) and (12.31) is

$$z^{d-1}B^{+}(z)B^{-*}(z)C(z) = 0$$

Thus the control law of (12.31) can be interpreted as a pole-placement controller, which gives this characteristic equation.

Multiplication of (12.32) by B^+ gives the equation

$$A(z)R(z) + B(z)S(z) = z^{d-1}B^{+}(z)B^{-*}(z)C(z) \tag{12.37}$$

where $R(z) = B^{+}(z)F(z)$ and $S(z) = G(z)$. This equation is the same Diophantine equation that was used in the pole-placement design [compare with Eq. (5.22)]. The closed-loop system has poles corresponding to the observer dynamics, to the stable process zeros, and to the reflections in the unit circle of the unstable process zeros. Notice that the transfer function $B(z)/A(z)$ may be interpreted as having $d = \deg A - \deg B$ zeros at infinity. The reflections of these zeros in the unit circle also appear as closed-loop poles, which are located at the origin.

Equation (12.37) shows that the closed-loop system is of order $2n - 1$ and that $d - 1$ of the poles are in the origin. A complete controller consisting of a full Kalman filter observer and feedback from the observed states gives a closed-loop system of order $2n$. The “missing” pole is due to a cancellation of a pole at the origin in the controller. This is further discussed in Sec. 12.5.

12.5 Linear Quadratic Gaussian (LQG) Control

The optimal control problem for the system of (12.5) with the criterion of (12.8) is now solved. The minimum-variance control law discussed in Sec. 12.4 can be expressed in terms of a solution to a polynomial equation. The solution to the LQG-problem can be obtained in a similar way. Two or three polynomial equations are needed, however. These equations are discussed before the main result is given.

The name *Gaussian* in LQG is actually slightly misleading. The proofs show that the probability distribution is immaterial as long as the random variables $e(k)$ are independent.

Using the state-space solution it is possible to get an interpretation of the properties of the optimal solution. These properties can be expressed in terms of the poles of the closed-loop system. In this way we can establish a connection between LQG design and pole placement.

Properties of the State-Space Solution

The problems discussed in this chapter was solved using state-space methods in Chapter 11. A state-space representation of the model of (12.5) is first given. For this purpose it is assumed that the model is normalized, so that $\deg C(z) = \deg A(z)$. The model of (12.5) can then be represented as

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) + Ke(k) \\ y(k) &= Cx(k) + e(k) \end{aligned}$$

where

$$\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} \quad K = \begin{pmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_{n-1} - a_{n-1} \\ c_n - a_n \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \tag{12.38}$$

Because this is an innovations representation if the matrix $\Phi - KC$ has all its eigenvalues inside the unit disc. The steady-state Kalman filter is then obtained by inspection:

$$\hat{x}(k+1 \mid k) = \Phi\hat{x}(k \mid k-1) + \Gamma u(k) + K\big(y(k) - C\hat{x}(k \mid k-1)\big) \tag{12.39}$$

The Kalman filter has the characteristic polynomial

$$\det\big(zI - (\Phi - KC)\big) = C(z) \tag{12.40}$$

This implies that $C(z)$ are some of the closed-loop poles. Assume a computational delay of one sampling period in the control law. The optimal control law is then

$$u(k) = -L\hat{x}(k \mid k-1)$$

and the transfer function of the controller is

$$H_r(z) = -L(zI - \Phi + KC + \Gamma L)^{-1}K = -\frac{S(z)}{R(z)} \tag{12.41}$$

where $R(z) = \det(zI - \Phi + KC + \Gamma L)$, $\deg R(z) = n$, and $\deg S(z) < n$. It follows from this discussion and Sec. 11.4 that the closed-loop poles are $C(z)$ and

$$P(z) = \det(zI - \Phi + \Gamma L)$$

where $P(z)$ is obtained from the algebraic Riccati equation.

It is more complicated to derive the control law when the admissible control is such that $u(k)$ is a function of $y(k), y(k-1), \ldots$. The loss function (12.8) corresponds to (11.9) with $Q_1 = C^TC$, $Q_{12} = 0$, and $Q_2 = \rho$. From (11.19) and (11.24) it follows that $L = L_v\Phi$. The results from state-space theory (Remark 2 of Theorem 11.7) show that the control law is

$$\begin{aligned} u(k) &= -L\hat{x}(k \mid k) - L_v\hat{v}(k \mid k) \\ &= -L\hat{x}(k \mid k) - L_vK\big(y(k) - C\hat{x}(k \mid k-1)\big) \\ &= -L_v(\Phi - KC)\hat{x}(k \mid k-1) - L_vKy(k) \end{aligned} \tag{12.42}$$

where $\hat{x}(k \mid k-1)$ is given by (12.39). The controller is still of order n. Eliminating $\hat{x}$ between (12.39) and (12.42), we find that the controller can be described by the relation

$$\begin{aligned} u(k) &= -L_v(\Phi - KC)(qI - \Phi + KC)^{-1}\Big(\Gamma u(k) + Ky(k)\Big) - L_vKy(k) \\ &= -L_v(\Phi - KC)(qI - \Phi + KC)^{-1}\Gamma u(k) \\ &\quad - L_v(\Phi - KC + qI - \Phi + KC)(qI - \Phi + KC)^{-1}Ky(k) \\ &= -L_v(\Phi - KC)(qI - \Phi + KC)^{-1}\Gamma u(k) \\ &\quad - L_vq(qI - \Phi + KC)^{-1}Ky(k) \end{aligned} \tag{12.43}$$

Introducing $R_2(q) = \det(qI - \Phi + KC)$ we get

$$u(k) = -\frac{R_1(q)}{R_2(q)}u(k) - \frac{S(q)}{R_2(q)}y(k)$$

where $\deg R_1(z) = n$, $\deg R_2(z) < n$ and $\deg S(z) = n$ with $S(0) = 0$. Hence

$$u(k) = -\frac{S(q)}{R_1(q) + R_2(q)}y(k) = -\frac{S(q)}{R(q)}y(k) \tag{12.44}$$

We thus find that the controller has the property $\deg R(z) = \deg S(z) = n$. Furthermore the condition $S(0) = 0$ implies that $\deg S^*(z) < n$.

Spectral Factorization

The LQ-problem is solved in Sec. 11.4 using the state-space approach, which led to a steady-state Riccati equation. It follows from the Riccati equation that

$$rP(z)P(z^{-1}) = \rho A(z)A(z^{-1}) + B(z)B(z^{-1}) \tag{12.45}$$

where the monic polynomial $P(z)$ is the characteristic polynomial of the closed-loop system. [see Eq. (11.40)]. The closed-loop characteristic polynomial can be obtained by solving a steady-state Riccati equation. An alternative is to find a polynomial $P(z)$ that satisfies (12.45) directly. A feedback that gives the desired closed-loop poles can then be determined by pole placement. The problem of finding a polynomial $P(z)$ that satisfies (12.45) is called *spectral factorization*.

First, consider a polynomial of the form

$$F(z) = f_0z^{2n} + f_1z^{2n-1} + \cdots + f_{n-1}z^{n+1} + f_nz^n + f_{n-1}z^{n-1} + \cdots + f_1z + f_0$$

Such a polynomial is self-reciprocal because

$$F^*(z) = z^{2n}F(z^{-1}) = F(z)$$

It then follows that if $z = a$ is a zero of $F(z)$, then $z = 1/a$ is also a zero. Moreover, if the coefficients f_i are real, then $z = \bar{a}$ and $z = 1/\bar{a}$ are also zeros, where $\bar{a}$ is the complex conjugate of a. The following result can now be established.

LEMMA 12.1 Let the real polynomials $A(z)$ and $B(z)$ be relatively prime with $\deg A(z) > \deg B(z)$. Then there exists a unique polynomial $P(z)$ with $\deg P(z) = \deg A(z) = n$ and all its zeros inside the unit disc or on the unit circle such that (12.45) holds. If $\rho > 0$, then $P(z)$ has no zeros on the unit circle.

Proof. A self-reciprocal polynomial is obtained if the right-hand side of (12.45) is multiplied by z^n. The zeros of the right-hand side are thus mirror images with respect to the unit circle. Because the coefficients are real, the zeros are also symmetric with respect to the real axis. The right-hand side of (12.45) cannot have zeros on the unit circle because if $z = e^{i\omega}$ is such a zero, then

$$\rho A(e^{i\omega})A(e^{-i\omega}) + B(e^{i\omega})B(e^{-i\omega}) = \rho|A(e^{i\omega})|^2 + |B(e^{i\omega})|^2 = 0$$

As $\rho > 0$, this implies that $z = \exp(i\omega)$ is a zero of both $A(z)$ and $B(z)$, which contradicts the assumption that $A(z)$ and $B(z)$ are relatively prime. The condition $\deg P(z) = n$ ensures a unique $P(z)$. ∎

Remark 1. By introducing reciprocal polynomials, Eq. (12.45) can be written as

$$rP(z)P^*(z) = \rho A(z)A^*(z) + z^d B(z)B^*(z) \tag{12.46}$$

where $P^*(z) = z^n P(z^{-1})$, and so on.

Remark 2. If $P(z)$ satisfies (12.45) so does $z^l P(z)$, where l is an arbitrary integer. To obtain a unique P we can either specify the degree of P or choose P as the polynomial of lowest degree that satisfies (12.45). For a control problem it is natural to interpret $P(z)$ as the closed loop characteristic polynomial under state feedback. With this interpretation it is natural to require that $\deg P(z) = \deg A(z) = n$. Notice that it is possible to find a P of lower degree when $\rho = 0$ or when $A(0) = 0$.

Conceptually the spectral-factorization problem can be solved by finding the zeros of the right-hand side of (12.45) and sorting them. There are also efficient recursive algorithms for solving the problem.

Heuristic Discussion

The LQG-problem will now be related to the pole-placement problem. We will first give the solution heuristically. A formal solution will be given later. First, recall that the pole-placement problem required specifications of the closed-loop characteristic polynomial, which were chosen as $A_c(z)A_o(z)$ when A_o was interpreted as the observer polynomial. In the LQG-problem the observer polynomial is simply $A_o(z) = C(z)$. Compare Theorem 12.1. The polynomial $A_c(z)$ is equal to the polynomial $P(z)$ obtained from the spectral factorization. When the polynomials $A_o(z) = C(z)$ and $A_c(z) = P(z)$ are specified we can now expect that the optimal control law is given by

$$u(k) = -\frac{S(q)}{R(q)}\, y(k)$$

where $R(z)$ and $S(z)$ are solutions to the Diophantine equation

$$A(z)R(z) + B(z)S(z) = P(z)C(z) \tag{12.47}$$

The structure of the admissible control laws is determined by the polynomials $R(z)$ and $S(z)$. To describe a control law such that $u(k)$ is a function of $y(k)$, $y(k-1)$, …, and $u(k-1)$, $u(k-2)$, …, that is, no delay in the controller, the polynomials $R(z)$ and $S(z)$ should have the same degree. To describe a control law such that $u(k)$ is a function of $y(k-1), y(k-2), \ldots$, and $u(k-1), u(k-2), \ldots$, that is, one sampling period delay in the controller, the pole excess of $S(z)/R(z)$ should be one. The complexity of the control law is determined by the orders of the polynomials $R(z)$ and $S(z)$.

There are many polynomials $R(z)$ and $S(z)$ that satisfy (12.47). Compare the discussion in Sec. 5.3. Among all choices we will determine solutions that minimize the loss function (12.8). Before making a formal solution we will discuss the problem heuristically.

The solution to the LQG problem based on the state space approach gives the additional constraints that have to be imposed on the solution to (12.47). Equation (12.41) gave a polynomial interpretation of the state space solution. The optimal controller was in fact characterized by the following conditions on the controller polynomials: $\deg R(z) = n$ and $\deg S(z) < n$. If A and B are relative prime the optimal LQG-controller is thus the unique solution to (12.47) with $\deg S(z) < \deg A(z)$.

The problem is more complicated when there is no delay in the controller. The transfer function of the optimal controller in this case was given by Eq. (12.44) with $\deg R(z) = \deg S(z) = n$, and and $\deg S^*(z) < n$. These conditions are more conveniently expressed using another version of the Diophantine equation (12.47). Assuming $\deg R(z) = \deg S(z) = n$, writing (12.47) with argument z^{-1} and multiplying it by z^{2n} we find that

$$A^*(z)R^*(z) + z^d B^*(z)S^*(z) = P^*(z)C^*(z) \tag{12.48}$$

where

$$d = \deg A(z) - \deg B(z)$$

If $\deg A^*(z) = n$ the optimal controller is then the unique solution to (12.48) with $\deg S^*(z) < \deg A^*(z)$. Notice, however, that this does not give the optimal solution when $\deg A^*(z) < n$, i.e. when $A(0) = 0$. This case will be discussed in the next section where we give a direct solution of the LQG problem with polynomial calculations.

Formal Proof

After the informal discussion we will now give a formal proof of the statements. For this purpose we will first prove a preliminary result.

LEMMA 12.2 Let the polynomial $P(z)$ be a solution to the spectral factorization problem (12.46) and let $A(z)$ be monic. Assume that the polynomials $A(z)$ and $B(z)$ do not have common roots outside the unit disc or on the unit circle; then there exists a unique solution to the equations

$$\begin{aligned} A^*(z)X(z) + rP(z)S^*(z) &= B(z)C^*(z) \\ z^dB^*(z)X(z) - rP(z)R^*(z) &= -\rho A(z)C^*(z) \end{aligned} \tag{12.49}$$

with $\deg X(z) < n$, $\deg R^*(z) \le n$ and $\deg S^*(z) < n$, where $n = \deg A(z)$.

Proof. First, assume that polynomial $P(z)$ has distinct zeros z_i. Since $P(z)$ is stable we have $|z_i| < 1$. The values $A^*(z_i)$ and $B^*(z_i)$ cannot vanish simultaneously because this would contradict the assumption that $A(z)$ and $B(z)$ do not have common unstable factors. Evaluating (12.49) for $z = z_i$ we get

$$\begin{aligned} A^*(z_i)X(z_i) &= B(z_i)C^*(z_i) \\ z_i^dB^*(z_i)X(z_i) &= -\rho A(z_i)C^*(z_i) \end{aligned} \tag{12.50}$$

If both $A^*(z_i)$ and $B^*(z_i)$ are different from zero, both equations give the same result, since it follows from (12.46) that

$$\frac{B(z_i)}{A^*(z_i)} = -\frac{\rho A(z_i)}{z_i^dB^*(z_i)}$$

If $A^*(z_i) = 0$ and $B^*(z_i) \neq 0$ it follows from (12.46) that $B(z_i) = 0$. Since $A(z)$ is monic it also follows that $A^*(0) = 1$. This implies that $|z_i| \neq 0$. The equation

$$A^*(z_i)X(z_i) = B(z_i)C^*(z_i)$$

is trivially satisfied and the solution to (12.50) is

$$X(z_i) = -\frac{\rho A(z_i)C^*(z_i)}{z_i^dB^*(z_i)}$$

A similar argument shows that $X(z_i)$ is unique also when $B^*(z_i) = 0$ and $A(z_i) \neq 0$. We can thus determine $\deg P$ values $X(z_i)$. Using Lagrange's interpolation formula the polynomial $X(z)$ of degree $\deg P - 1$ which satisfies (12.50) is thus unique.

It follows from the construction of the polynomial $X(z)$ that the polynomial $A^*(z)X(z) - B(z)C^*(z)$ vanishes for the zeros z_i of $P(z)$. This implies that it is divisible by $P(z)$. The quotient

$$S^*(z) = \frac{A^*(z)X(z) - B(z)C^*(z)}{rP(z)}$$

is thus a polynomial. It has degree

$$\deg S^* \le \max(\deg A^* + \deg P - 1, \deg B + \deg C^*) - \deg P < n \tag{12.51}$$

Using the same argument we also find that

$$R^*(z) = \frac{z^dB^*(z)X(z) + \rho A(z)C^*(z)}{rP(z)}$$

is a polynomial of degree

$$\deg R^* \le \max(d + \deg B^* + \deg P - 1, \deg A + \deg C^*) - \deg P \le n \tag{12.52}$$

The solution $X(z)$, $S^*(z)$ and $R^*(z)$ to (12.49) is continuous in the coefficients of polynomials $A(z)$ and $B(z)$. If polynomial $P(z)$ has multiple zeros we can the perturb the coefficients of $A(z)$ and $B(z)$ to obtain a $P(z)$ with distinct zeros and obtain the results by a limiting procedure. The details of this argument are delicate. ∎

Remark 1. Notice that if one solution, X_0, R_0^*, S_0^*, to Eq. (12.49) has been obtained all other solutions are given by

$$\begin{aligned} X(z) &= X_0(z) + Q(z)rP(z) \\ R^*(z) &= R_0^*(z) + Q(z)z^dB^*(z) \\ S^*(z) &= S_0^*(z) - Q(z)A^*(z) \end{aligned} \tag{12.53}$$

where $Q(z)$ is an arbitrary polynomial. This is easily verified by direct insertion into the equation.

Remark 2. The polynomials $R(z)$ and $S(z)$ are given by $R(z) = z^nR^*(z^{-1})$ and $S(z) = z^nS^*(z^{-1})$. The conditions $A^*(0) = P^*(0) = C^*(0) = 1$ together with Eq. (12.53) imply that $R^*(0) = 1$, hence $\deg R(z) = n$ and $\deg S(z) \le n$ and $\deg S^*(z) < n$.

Remark 3. Eliminating X by multiplying the first equation by $z^dB^*(z)$ and the second by $A^*(z)$ and subtracting gives

$$rPS^*z^dB^* + rPA^*R^* = BC^*z^dB^* + \rho A^*C^* = rPP^*C^*$$

where the second equality follows from (12.46). Dividing by rP shows that the the polynomials R^* and S^* satisfy the Diophantine equation (12.48).

Remark 4. In the following we will need another property of the solutions to Eq. (12.49). Adding the first equation multiplied by ρA and the second multiplied by B gives:

$$(\rho AA^* + z^dBB^*)X + \rho rAPS^* - rPBR^* = 0$$

Using the spectral factorization condition (12.46) and dividing by rP now gives:

$$P^*(z)X(z) = B(z)R^*(z) - \rho A(z)S^*(z) \tag{12.54}$$

After these preliminaries we will now solve the LQG-problem with polynomial calculations.

THEOREM 12.4 LINEAR QUADRATIC GAUSSIAN CONTROL Consider the system in (12.5) with $\deg A(z) = \deg C(z) = n$. Assume that all the zeros of polynomial $C(z)$ are inside the unit disc, that there are no factors common to all three of the polynomials $A(z)$, $B(z)$, and $C(z)$, and that a possible common factor of $A(z)$ and $B(z)$ has all its zeros inside the unit disc. Let the monic polynomial $P(z)$, which has all its zeros inside the unit disc, be the solution to (12.45) with $\deg P(z) = n$. The admissible control law with no delay that minimizes the criterion of (12.8) is given by

$$u(k) = -\frac{S^*(q^{-1})}{R^*(q^{-1})}\, y(k) = -\frac{S(q)}{R(q)}\, y(k) \tag{12.55}$$

where polynomials $R^*(z)$ and $S^*(z)$ are the unique solution to Equation (12.49) with $\deg X(z) < n$. With the control law of (12.55), the output becomes

$$y(k) = \frac{R(q)}{P(q)}\, e(k) \tag{12.56}$$

and the control signal is

$$u(k) = -\frac{S(q)}{P(q)}\, e(k) \tag{12.57}$$

The minimal value of the loss function is

$$\min \mathrm{E}(y^2 + \rho u^2) = \frac{\sigma^2}{2\pi i} \oint \frac{R(z)R(z^{-1}) + \rho S(z)S(z^{-1})}{P(z)P(z^{-1})}\, \frac{dz}{z} \tag{12.58}$$

Proof. Introduce

$$u = v - \frac{S}{R}\, y \tag{12.59}$$

where v may be regarded as a transformed control variable, which has to be determined. Equations (12.5), (12.47), and (12.59) give

$$y = \frac{BRv + CRe}{AR + BS} = \frac{BRv + CRe}{PC} = \frac{BR}{PC}\, v + \frac{R}{P}\, e \tag{12.60}$$

It then follows from (12.60) that

$$u = v - \frac{SBv + SCe}{PC} = \frac{PC - BS}{PC}\, v - \frac{S}{P}\, e = \frac{AR}{PC}\, v - \frac{S}{P}\, e \tag{12.61}$$

The loss function of (12.8) can be written as

$$\begin{aligned} J &= \mathrm{E}(y^2 + \rho u^2) = \mathrm{E}\left(\frac{BR}{PC}\, v + \frac{R}{P}\, e\right)^2 + \rho \mathrm{E}\left(\frac{AR}{PC}\, v - \frac{S}{P}\, e\right)^2 \\ &= J_1 + 2J_2 + J_3 \end{aligned}$$

where

$$\begin{aligned} J_1 &= \mathrm{E}\left(\left(\frac{BR}{PC}\, v\right)^2 + \rho\left(\frac{AR}{PC}\, v\right)^2\right) \\ J_2 &= \mathrm{E}\left(\left(\frac{BR}{PC}\, v\right)\left(\frac{R}{P}\, e\right) - \rho\left(\frac{AR}{PC}\, v\right)\left(\frac{S}{P}\, e\right)\right) \\ J_3 &= \mathrm{E}\left(\left(\frac{R}{P}\, e\right)^2 + \rho\left(\frac{S}{P}\, e\right)^2\right) \end{aligned}$$

It follows from Remark 2 of Theorem 10.2 and (12.45) that

$$\begin{aligned} J_1 &= \frac{1}{2\pi i} \oint \frac{\left(B(z)B(z^{-1}) + \rho A(z)A(z^{-1})\right)R(z)R(z^{-1})}{P(z)P(z^{-1})C(z)C(z^{-1})}\, V(z)V(z^{-1})\, \frac{dz}{z} \\ &= \frac{r}{2\pi i} \oint \frac{R(z)R(z^{-1})}{C(z)C(z^{-1})}\, V(z)V(z^{-1})\, \frac{dz}{z} = r\mathrm{E}\left(\frac{R(q)}{C(q)}\, v\right)^2 \end{aligned}$$

For causal controllers with no time delay $v(t)$ can be expressed as $v(t) = V(q)e(t)$, where $V(q)$ is a rational function with zero pole excess.

$$J_2 = \frac{\sigma^2}{2\pi i} \oint \frac{B(z)R(z)R(z^{-1}) - \rho A(z)R(z)S(z^{-1})}{P(z)C(z)P(z^{-1})}\, V(z)\, \frac{dz}{z}$$

It follows from Equation (12.54) that

$$B(z)R(z^{-1}) - \rho A(z)S(z^{-1}) = P(z^{-1})X(z)$$

Hence

$$J_2 = \frac{\sigma^2}{2\pi i} \oint \frac{R(z)X(z)}{P(z)C(z)}\, V(z)\, \frac{dz}{z} = \mathrm{E}\left(\left(\frac{R(q)X(q)}{P(q)C(q)}\, v(k)\right) e(k)\right)$$

It was assumed that $P(z)$ and $C(z)$ are stable and it follows from Lemma 12.2 that $\deg X(z) < n$. This implies that

$$\deg R(z)X(z) < \deg P(z)C(z) = 2n$$

The quantity

$$\frac{R(q)X(q)}{P(q)C(q)}\, v(k)$$

is thus a function of $v(k-1), v(k-2), \ldots$. Because all these terms are independent of $e(k)$, J_2 becomes zero. The loss function can thus be written as

$$J = r\mathrm{E}\left(\frac{R(q)}{C(q)}\, v(k)\right)^2 + \mathrm{E}\left(\frac{R(q)}{P(q)}\, e(k)\right)^2 + \rho \mathrm{E}\left(\frac{S(q)}{P(q)}\, e(k)\right)^2$$

where P and C are stable polynomials. It follows that the loss function achieves its minimum (12.58) for $v = 0$, which by (12.59) corresponds to the control law of (12.55). Equations (12.56) and (12.57) follow from (12.60) and (12.61), and Theorem 10.2 and (10.23) give the formula of (12.58). ■

Remark 1. The minimum-variance control law is a special case of Theorem 12.4 with $\rho = 0$. It follows from (12.49) that $R^*(z)P(z) = -z^d B^*(z)X(z)$. Because $\deg X(z) < n$, we have $\deg R^*(z) < n$ for $\rho = 0$. Because also $\deg S^*(z) < n$ the polynomials $R(z)$ and $S(z)$ have z as a common factor. Introducing $B(z) = B^+(z)B^-(z)$, where B^+ has all its zeros inside the unit disc and B^- all its zeros outside the unit disc, we get

$$\sqrt{r}\, P(z) = z^d B^+(z)B^{-*}(z)$$

where $\sqrt{r} = B^-(0)$. The Diophantine equation (12.47) then becomes

$$A(z)R(z) + B(z)S(z) = z^d B^+(z)B^{-*}(z)C(z)/\sqrt{r}$$

Cancelling the common factor z in $R(z)$ and $S(z)$ to give $R(z)$ and $S(z)$ we get

$$A(z)\tilde{R}(z) + B(z)\tilde{S}(z) = z^{d-1}B^+(z)B^{-*}(z)C(z)/\sqrt{r}$$

which is identical to (12.32). Theorem 12.3 has thus been proven in a different way. The pole-zero cancellation at the origin of the control law explains that there are $d-1$ instead of d closed-loop poles at the origin. Compare with (12.29).

Remark 2. If the polynomial $A(z)$ has the form $A(z) = z^l A_1(z)$, where $l \le d = \deg A(z) - \deg B(z)$, it follows from (12.45) that $P(z) = z^l P_1(z)$. Equation (12.47) then implies that $S(z) = z^l S_1(z)$.

The LQG controller will now be illustrated by an example.

Example 12.10 LQG control with unstable process zero

Consider the same system as in Examples 12.8 and 12.9. Instead of using a minimum-variance control law we will now use an LQG strategy. To do this the parameter ρ in the control strategy must be chosen. To guide this choice we will first calculate the variances of the output and control signals obtained for different values of the loss function. The results are shown in Fig. 12.8. The value $\rho = 0$ corresponds to a minimum-variance strategy. This gives a control signal with large variance. Compare with Example 12.9. The variance of the control signal decreases rapidly with increasing ρ. The variance of the output increases slowly.

By choosing a reasonable value of ρ it is possible to have a control strategy that gives an output variance that is only marginally higher than with minimum-variance control and a variance of the control signal that is substantially lower. A reasonable value is $\rho = 1$. This gives $\mathrm{E}y^2 = 1.39$ and $\mathrm{E}u^2 = 0.22$, which can be compared with minimum-variance control that gives $\mathrm{E}y^2 = 1.05$ and $\mathrm{E}u^2 = 14.47$.

The input- and output signals obtained with $\rho = 1$ are shown in Fig. 12.9. Compare with the corresponding curves for minimum-variance control in Example 12.9. The fluctuations in the output are a little larger, but the fluctuations in the control signal are substantially smaller. This way of applying LQG control where the control weighting is used as a design parameter is very typical. ■

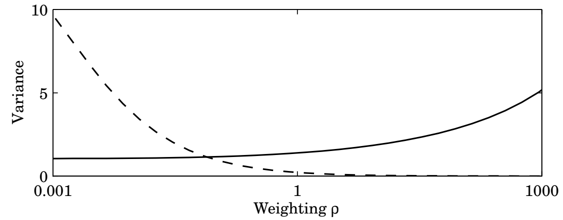

Figure 12.8 Variances of input u (dashed line) and output y (solid line) for LQG controllers having different values of the control weighting ρ for the system in Example 12.10

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An Interpretation

Theorem 12.4 establishes the relation between LQG-control and pole-placement control because the polynomial $C(z)$ is the observer polynomial $A_o(z)$ and $P(z)$ is the polynomial $A_c(z)$. The LQG-controller may thus be considered as a pole-placement controller where the observer polynomial $A_o(z)$ is obtained from the noise characteristics and the polynomial $A_c(z)$ from the solution to an optimization problem. The solution to the optimization problem also tells what solution of the Diophantine equation we should choose.

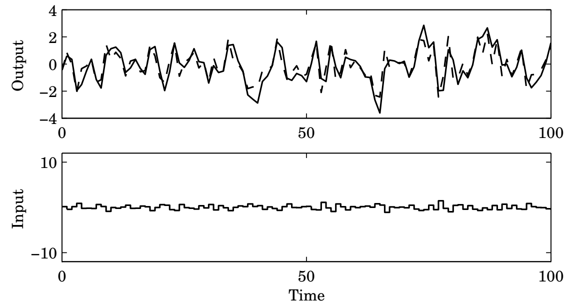

Figure 12.9 Simulation of the for the system in Example 12.10 using the LQG-controller with $\rho = 1$. The output obtained with the minimum-variance controller ($\rho = 0$) is shown in dashed. Also compare with Fig. 12.6.

A Computational Procedure

Theorem 12.4 gives a convenient way to compute the LQG-control law for SISO systems, which can be described as follows.

1. Rewrite the model of the process and the disturbance in the standard form (12.5), where $C(z)$ is a stable polynomial. It may be necessary to use a spectral factorization to obtain this form.

2. Use a spectral factorization to calculate $P(z)$. If the polynomials $A(z)$ and $B(z)$ have a stable common factor $A_2(z)$, the calculations of the control law can be simplified by first factoring $A(z)$ and $B(z)$ as $A(z) = A_1(z)A_2(z)$ and $B(z) = B_1(z)A_2(z)$. It follows from (12.45) that $A_2(z)$ also divides $P(z)$. This polynomial can thus be written as $P(z) = P_1(z)A_2(z)$, where $P_1(z)$ is given by

$$rP_1(z)P_1(z^{-1}) = \rho A_1(z)A_1(z^{-1}) + B_1(z)B_1(z^{-1})$$

The polynomial $P(z)$ is then equal to $P_1(z)A_2(z)$, which is stable, because $A_2(z)$ was assumed stable. Equation (12.47) can also be divided by $A_2(z)$ to give

$$P_1(z)C(z) = A_1(z)R(z) + B_1(z)S(z)$$

where $\deg R(z) = \deg S(z) = \deg C(z) = n$, and $S(0) = 0$.

3a. If there are no common factors between A and B and if $A(0) \neq 0$ then the controller is given by a unique solution to the Diophantine equation (12.47) such that $\deg R(z) = \deg S(z) = n$, and $S(0) = 0$.

3b. If there are stable common factors of A and B or if $A(0) = 0$ the solution is obtained from the Equation (12.49) or Diophantine equation (12.47), and (12.54).

The computational procedure shows that when there are no common factors between A and B and when $A(0) \neq 0$ then it is sufficient to solve only one Diophantine equation with the extra constraint $S(0) = 0$ to obtain a unique solution. In other cases it is necessary to solve the coupled equations (12.49). Theorem 12.4 is illustrated by two examples.

Example 12.11 LQG for first order system

Consider a system characterized by

$$\begin{aligned} A(z) &= z + a \qquad a \neq 0 \\ B(z) &= b \\ C(z) &= z + c \end{aligned}$$

To find the control law that minimizes the criterion of (12.8), the spectral-factorization problem is first solved. Equation (12.45) can be written as

$$r(z + p_1)(z^{-1} + p_1) = \rho(z + a)(z^{-1} + a) + b^2$$

Equating coefficients of equal powers of z gives

$$\begin{aligned} rp_1 &= \rho a \\ r(1 + p_1^2) &= \rho(1 + a^2) + b^2 \end{aligned}$$

Elimination of p_1 gives

$$r^2 - r\big(\rho(1 + a^2) + b^2\big) + \rho^2 a^2 = 0 \tag{12.62}$$

This equation has the solution

$$r = \frac{1}{2}\left(\rho(1 + a^2) + b^2 + \sqrt{\rho^2(1 - a^2)^2 + 2\rho b^2(1 + a^2) + b^4}\right)$$

where the positive root is chosen to give $|p_1| < 1$. Furthermore

$$p_1 = \frac{\rho a}{r}$$

Because A and B are relative prime and $A(0) \neq 0$, the solution can be found from the Diophantine equation (12.47). With $\deg S = 1$ and $S(0) = 0$, Eq. (12.47) becomes

$$(z + a)(z + r_1) + bs_0 z = (z + p_1)(z + c)$$

Putting $z = -a$ we get

$$s_0 = -\frac{(p_1 - a)(c - a)}{ab}$$

It follows from (12.62) that

$$\rho a p_1^2 - \rho p_1(1 + a^2) - p_1 b^2 + \rho a = 0$$

Hence

$$\rho(a p_1^2 - a^2 p_1 - p_1 + a) = p_1 b^2$$

or

$$\rho\Big(a p_1(p_1 - a) - (p_1 - a)\Big) = p_1 b^2$$

which gives

$$p_1 - a = -\frac{p_1 b^2}{\rho(1 - a p_1)}$$

We thus get

$$s_0 = \frac{p_1 b^2(c - a)}{\rho a b(1 - a p_1)} = \frac{b(c - a)}{r(1 - a p_1)}$$

Furthermore, equating the constant terms in (12.47) gives

$$r_1 = \frac{p_1 c}{a} = \frac{\rho c}{r}$$

The control law thus becomes

$$u(k) = -\frac{S(q)}{R(q)}\, y(k) = -\frac{b(c - a)}{r(1 - a p_1)}\, \frac{q}{q + p_1 c/a}\, y(k)$$

■

The calculations in Example 12.11 do not work when $a = 0$, because in this case the solution to the LQG-problem is not uniquely determined by the Diophantine equation (12.47) and it is necessary to use (12.49).

Example 12.12 LQG for system with a time-delay

Consider the case

$$A(z) = z$$
$$B(z) = b$$
$$C(z) = z + c$$

The spectral factorization problem (12.45) has the solution

$$P(z) = z \qquad r = \rho + b^2$$

Assuming that it is desired to have a controller with no extra delay we require that $\deg S(z) = \deg R(z) = 1$. The Diophantine equation (12.47) with the constraint $\deg S^*(z) = 0$ becomes

$$z(z + r_1) + bs_0 z = z(z + c)$$

Identification of coefficients of equal power of z gives only one equation

$$r_1 + bs_0 = c$$

to determine two parameters r_1 and s_0. The approach with the Diophantine equation thus does not work in this case. Equation (12.49) gives

$$x_0 + rs_0 z = b(1 + cz)$$
$$bx_0 z - rz(1 + r_1 z) = -\rho z(1 + cz)$$

Identification of coefficients of equal power of z gives linear equations which have the solution

$$x_0 = b$$
$$r_1 = \frac{\rho c}{r} = \frac{\rho c}{\rho + b^2}$$
$$s_0 = \frac{bc}{r}$$

■

Uncontrollable and Unstable Modes

Models with the property that polynomials $A(z)$ and $B(z)$ have a common factor that is not a factor of $C(z)$ are important in practice. They appear when there are modes that are excited by disturbances and uncontrollable from the input. Compare Sec. 12.2. Because the modes are not controllable, they are not influenced by feedback.

Theorem 12.4 covers the case of stable common factors, but it does not work for unstable common factors. Unstable common factors are important in practice because they give one way of obtaining regulators with integral action.

To see what happens when there are unstable common factors, let A_2 denote the greatest common divisor of A and B and let A_2^- denote the factor of A_2 with zeros outside the unit disc or on the unit circle. Let the feedback be

$$u(k) = -\frac{S(q)}{R(q)}\, y(k)$$

where $R(z)$ and $S(z)$ are relatively prime. It follows from (12.5) that

$$y(k) = \frac{R(q)C(q)}{A(q)R(q) + B(q)S(q)}\, e(k) \qquad (12.63)$$

$$u(k) = -\frac{S(q)C(q)}{A(q)R(q) + B(q)S(q)}\, e(k) \qquad (12.64)$$

The unstable factor $A_2^-(z)$ divides the denominators of the right-hand sides of (12.63) and (12.64). Both y and u will be unbounded unless $R(z)$ or $S(z)$ are chosen in special ways. The signal y will be bounded if $R(z)$ is divisible by $A_2^-(z)$, and u will be bounded if $A_2^-(z)$ divides $S(z)$. Because $R(z)$ and $S(z)$ are relatively prime, it is not possible to make both y and u bounded. This is natural because infinitely large control actions are necessary to compensate for infinitely large disturbances.

To describe a problem of this type as a meaningful optimization problem, the criterion of (12.8) must be modified. One possibility is to introduce the variable

$$w(k) = q^{-m} A_2^-(q) u(k) \qquad (12.65)$$

where $m = \deg A_2^-(z)$, and to introduce the criterion

$$J_{lq}' = \mathrm{E}\left(y^2(k) + \rho w^2(k)\right) \qquad (12.66)$$

Example 12.13 Integral action

Let the system be described by

$$y(k) = \frac{B_1(q)}{A_1(q)}\, u(k) + \frac{C_1(q)}{q - 1}\, e(k)$$

which is a special case of Eqs. (12.1) to (12.4) with a drifting disturbance. Hence

$$A(q) = (q - 1)A_1(q)$$
$$B(q) = (q - 1)B_1(q)$$
$$C(q) = A_1(q)C_1(q)$$

Unbounded control signals are necessary to compensate for the unbounded disturbance. This implies that the modified loss function (12.66) becomes

$$J_{lq}' = E\left[y^2(k) + \rho\Big(\Delta u(k)\Big)^2\right]$$

where

$$\Delta u(k) = u(k) - u(k - 1)$$

This means that the difference and not the absolute value of the control signal is penalized. The solution to the LQG-problem gives a controller with integral action.

■

The following result can then be established.

THEOREM 12.5 LQG-CONTROL WITH UNSTABLE COMMON FACTORS Consider the system described by (12.5), where $A(z)$ and $C(z)$ are monic polynomials of degree n. Assume that all zeros of $C(z)$ are inside the unit disc and that there is no nontrivial polynomial that divides $A(z)$, $B(z)$, and $C(z)$. Let $A_2(z)$ be the greatest common divisor of $A(z)$ and $B(z)$, let $A_2^+(z)$ of degree l be the factor of $A_2(z)$ with all its zeros inside the unit disc, and let $A_2^-(z)$ of degree m be the factor of $A(z)$ that has zeros on the unit circle or outside the unit disc. The admissible control law that minimizes (12.66) is given by

$$u(k) = -\frac{S(q)}{R(q)}\, y(k)$$

where $R(z)$ and $S(z)$ are of degree $n+m$

$$\begin{aligned} R(z) &= A_2^-(z)\tilde{R}(z) \\ S(z) &= z^m \tilde{S}(z) \end{aligned} \tag{12.67}$$

and $\tilde{R}(z)$ and $\tilde{S}(z)$ satisfies

$$\begin{aligned} A_1(z)A_2^-(z)\tilde{R}(z) + z^m B_1(z)\tilde{S}(z) &= P_1(z)C(z) \\ A^*(z)X(z) + rP(z)\tilde{S}^*(z) &= q^m \tilde{B}(z)C^*(z) \end{aligned} \tag{12.68}$$

with $\deg \tilde{R}(z) = \deg \tilde{S}(z) = n$, $\deg X(z) < n$ and $\tilde{S}(0) = 0$. Furthermore

$$\begin{aligned} A(z) &= A_1(z)A_2(z) \\ B(z) &= B_1(z)A_2(z) \\ \tilde{B}(z) &= B_1(z)A_2^+(z) \end{aligned}$$

and $P_1(z)$ is the solution of the spectral-factorization problem

$$rP_1(z)P_1(z^{-1}) = \rho A_1(z)A_2^-(z)A_1(z^{-1})A_2^-(z^{-1}) + B_1(z)B_1(z^{-1}) \tag{12.69}$$

with $\deg P_1(z) = \deg A_1(z) + \deg A_2^-(z)$.

Proof. Introducing the signal (12.65), the model (12.5) can be written as

$$A(q)y(k) = \tilde{B}(q)q^m w(k) + C(q)e(k)$$

The polynomials $A(q)$ and $\tilde{B}(q)$ have the common factor $A_2^+(z)$, which has all its zeros inside the unit disc, but no other common factors with zeros outside the unit disc or on the unit circle. It then follows from Theorem 12.4 that the optimal control law

$$w(k) = -\frac{\tilde{S}(q)}{\tilde{R}(q)}\, y(k)$$

is obtained from (12.47). Because $A(z)$ and $\tilde{B}(z)$ have the stable common factor $A_2^+(z)$, the polynomial $P(z)$ has the form

$$P(z) = A_2^+(z)P_1(z)$$

where $P_1(z)$ is the solution to the spectral-factorization problem (12.69). From Lemma 12.2 the polynomials $\tilde{R}(z)$ and $\tilde{S}(z)$ satisfy the equations

$$\begin{aligned} A(z)\tilde{R}(z) + z^m \tilde{B}(z)\tilde{S}(z) &= A_2^+(z)P_1(z)C(z) \\ A^*(z)X(z) + rP(z)\tilde{S}^*(z) &= q^m \tilde{B}(z)C^*(z) \end{aligned}$$

with $\deg \tilde{R}(z) = \deg \tilde{S}(z) = n$. Because A_2^+ divides $A(z)$ and $\tilde{B}(z)$ we get (12.68). Using (12.65) to express the control law in terms of the control variable u gives the result. ■

Remark. Notice that using (12.67), Eq. (12.68) can be written as

$$A(z)R(z) + B(z)S(z) = A_2(z)P_1(z)C(z)$$

The LQG-solution can thus be interpreted as a pole-placement controller, where the poles are positioned at the zeros of A_2, P_1, and C. The controller also has the property that A_2^- divides R. This is an example of the internal model principle.

Command Signals

The discussion in this chapter has so far been limited to the regulator problem. To introduce command signals, refer to the discussion in Chapter 5. The key issue is to introduce the command signals in such a way that they do not generate unnecessary reconstruction errors. This is achieved by the control law

$$R(q)u(k) = t_0 A_o(q)u_c(k) - S(q)y(k)$$

where $A_o(q)$ is the observer polynomial and to a constant. For the optimal Kalman filter $A_o(q) = C(q)$, where $C(q)$ is given by (12.40). It then follows from (12.5) that the output of the system is given by

$$y(k) = t_0 \frac{B(q)}{P(q)}\, u_c(k) + \frac{R(q)}{P(q)}\, e(k)$$

where $\deg R = n$.

The pulse-transfer function from the command signal is $B(z)/P(z)$. This response may be shaped further by cascading with a precompensator that has an arbitrary stable transfer function $H_f(z)$. The control law becomes

$$u(k) = \frac{A_o(q)}{R(q)}\, H_f(q)u_c(k) - \frac{S(q)}{P(q)}\, y(k)$$

which gives

$$y(k) = \frac{B(q)}{P(q)} H_f(q) u_c(k) + \frac{R(q)}{P(q)} e(k)$$

Because the polynomial P is stable, this may be canceled by the precompensator. It thus follows that the response for disturbances and command signals may be shaped differently.

The feedback S/R is first designed to ensure a good response to disturbances. The precompensator H_f is then chosen to obtain the desired response to command signals.

12.6 Practical Aspects

Much of the arbitrariness of design seems to disappear when design problems are formulated as optimization problems. The model and the criteria are stated, and the control law is obtained simply as the solution to an optimization problem. This simplicity is deceptive because the arbitrariness is instead transferred to the modeling and the formulation of criteria. A successful application of optimization theory requires insight into how the properties of the model and the criteria are reflected in the control law. Typical questions are: What should the model look like in order to get a regulator with integral action? What problem statements give regulators with a PID-structure? Some of these issues are discussed in this section, which also gives insight into the properties of the optimal control laws. It turns out that some results can be formulated as design rules. The polynomial approach, which operates directly with the transfer functions, is well suited to do this.

Other aspects of practical relevance, such as sensitivity and robustness, are also discussed. A brief treatment of the intersample ripple of the loss function is given, together with some aspects of the choice of the sampling period.

Properties of the Optimal Regulator

Some properties of the model influence the optimal-control laws. The basic model used is given by (12.5)—that is,

$$A(q)y(k) = B(q)u(k) + C(q)e(k) \tag{12.70}$$

The ratio B/A represents the pulse-transfer function of the process, and the ratio C/A represents the pulse-transfer function that generates the disturbance of the process output. The polynomials A, B, and C may have common factors that reflect the way the control signal and the disturbance are coupled to the system. There are, however, no factors common to all three polynomials. Compare this with the discussion in Sec. 12.2, where the model is derived. The presence of common factors that will directly influence the properties of the regulators will now be investigated.

The internal-model principle. Factors that are common to polynomials A and B correspond to disturbance modes that are not controllable from u. Such modes will appear as factors of P. Let

$$A_2 = \gcd(A, B)$$

be the greatest common divisor of polynomials A and B. If A_2 is stable, it follows from Theorem 12.4 that A_2 also divides P. If A_2 has a factor A_2^- with all its zeros outside the unit disc, the corresponding result follows from Theorem 12.5. In this case it also follows from Theorem 12.5 that A_2 divides R. This observation is called the *internal-model principle*; it says that to regulate a system with unstable disturbances, the disturbance dynamics must also appear in the dynamics of the regulator. A few examples illustrate this idea.

Example 12.14 Integral action

A regulator has integral action if $z-1$ divides $R(z)$. It follows from Theorem 12.5, and the internal-model principle, that this will occur if $z-1$ divides both A and B, which means that the model is of the form

$$A_1(q)(q-1)y(k) = B_1(q)(q-1)u(k) + C(q)e(k)$$

This means that there is a drifting disturbance. ■

Example 12.15 Elimination of a sinusoidal disturbance

A narrow-band sinusoidal disturbance with frequency centered at ω may be represented as white noise driving a system with the denominator

$$D(q) = q^2 - 2q\cos\omega h + 1$$

If the poles of the system dynamics do not correspond to D, the model becomes

$$A_1(q)D(q)y(k) = B_1(q)D(q)u(k) + C(q)e(k)$$

The optimal regulator is then such that $D(z)$ divides $R(z)$. ■

Cancellation of process poles. A common factor of A and C corresponds to controllable modes that are not excited by the disturbances. Let A_2 be the greatest common divisor of A and C. The polynomial A_2 is stable because C is stable, and it does not divide B because there is no factor that divides all of A, B, and C. It follows from (12.47) that A_2 also divides the polynomial S, which is the numerator of the regulator transfer function. Thus *stable process poles that are not excited by the disturbances may be canceled*.

Cancellation of process zeros. Common factors of B and C correspond to process zeros that block transmission both for the control signal u and for the disturbance e. Let B_2 be the greatest common divisor of B and C. The polynomial B_2 is stable and it does not divide A. It then follows from (12.47) that B_2 divides R. This means that the zeros corresponding to $B_2 = 0$ are canceled by the regulator. Therefore, *process zeros that are also transmission zeros for the disturbance C are canceled by the regulator*.

For the minimum-variance control, it follows from (12.46) with $\rho = 0$ that

$$\sqrt{r}P = q^d B^+ B^{-*}$$

where $\sqrt{r} = B^-(0)$ and from (12.47) that B^+ divides R. All stable zeros are thus canceled by the minimum-variance control law.

An analysis of the properties of the optimal-control law thus gives partial answers to the classic cancellation problem.

Sensitivity and Robustness

It is important that a control system be insensitive or robust with respect to measurement errors, plant disturbances, and modeling errors. This may be analyzed as in Sec. 5.5 for the pole-placement problem. The robustness properties are conveniently expressed in terms of the loop gain:

$$\mathcal{L} = \frac{BS}{AR}$$

or the return difference

$$H_{rd} = \frac{1}{\mathcal{S}} = 1 + \frac{BS}{AR} = \frac{AR + BS}{AR} = \frac{PC}{AR}$$

The loop gain $\mathcal{L}(\exp i\omega h)$ is normally high for low frequencies and small for high frequencies. The crossover frequency ω_c is the lowest frequency, where

$$\left|\mathcal{L}\left(e^{i\omega_c h}\right)\right| = 1$$

The closed-loop system is insensitive to plant disturbances at those frequencies where the loop gain is high. To have low sensitivity to poor modeling of the high-frequency dynamics of the plant, it is desirable that the loop gain decreases rapidly above the crossover frequency. It is possible to make sure that the loop gain is high for certain frequencies by choosing models with special structure, as was done in Examples 12.14 and 12.15. Plots similar to those in Fig. 5.6 are also useful in evaluating the sensitivity. In a properly designed sample-data system, there will be antialiasing filters, which eliminate signal transmission above the Nyquist frequency. The selection of a proper sampling rate is one way to make sure that the loop gain is low over a given frequency. This also means that high-frequency modeling errors have little influence. Notice, however, that plots of the loop gain and the return difference will not give the complete picture because there may be pole-zero cancellations that do not show up in these plots.

An analysis of the characteristic equations is useful in such a case. To perform such an analysis, assume that the system is governed by

$$A^0(q)y(k) = B^0(q)u(k) + C^0(q)e(k) \tag{12.71}$$

but that a regulator is designed based on a different model, as in (12.70). The regulators given by Theorems 12.4 and 12.5 give a closed-loop system with the characteristic polynomial

$$\begin{aligned} A^0R + B^0S &= A^0R - AR + B^0S - BS + AR + BS \\ &= PC + (A^0 - A)R + (B^0 - B)S \end{aligned}$$

When the model of (12.70) is equal to the system of (12.71) the characteristic polynomial is $PC = P_1A_2C$, as expected. By continuity it also follows that small changes in the system give small changes in the closed-loop poles. The system is sensitive to changes in the parameters if polynomial P_1 or C have zeros close to the unit circle.

To guarantee systems with a low sensitivity, it is necessary to impose further constraints. Recall that both C and P were obtained as solutions to a spectral-factorization problem.

Closed-Loop Systems with Guaranteed Exponential Stability

The control laws given by Theorems 12.2, 12.3, 12.4, and 12.5 give closed-loop systems with poles inside the unit disc. It is sometimes desirable to have control laws such that the closed-loop system has its poles inside a circle with radius $\bar{r}$. It is straightforward to formulate optimization problems that give such control laws.

Introduce the criterion

$$J = \mathrm{E}\bar{r}^{-2k}\left(y^2(k) + \rho u^2(k)\right) \tag{12.72}$$

If a control law that minimizes this criterion can be found, the variables $y(k)$ and $u(k)$ must converge to zero at least as fast as $\bar{r}^k$ when k increases. To obtain such a result, it must be assumed that the model of (12.5) is such that the covariance of $e(k)$ also goes to zero as $\bar{r}^k$.

Introduce the scaled variables η, μ, and ε defined by

$$\begin{aligned} y(k) &= \bar{r}^k\eta(k) \\ u(k) &= \bar{r}^k\mu(k) \\ e(k) &= \bar{r}^k\varepsilon(k) \end{aligned}$$

Because

$$q^l y(k) = q^l\left(\bar{r}^k\eta(k)\right) = \bar{r}^{k+l}\eta(k+l) = \bar{r}^k(\bar{r}q)^l\eta(k)$$

it follows that

$$A(q)y(k) = A(q)\left(\bar{r}^k \eta(k)\right) = \bar{r}^k A(\bar{r}q)\eta(k)$$

Introducing the transformed polynomials

$$\tilde{A}(z) = A(\bar{r}z)$$
$$\tilde{B}(z) = B(\bar{r}z)$$
$$\tilde{C}(z) = C(\bar{r}z)$$

the model of (12.5) can be written as

$$\tilde{A}(q)\eta(k) = \tilde{B}(q)\eta(k) + \tilde{C}(q)\varepsilon(k) \tag{12.73}$$

and the criterion of (12.72) becomes

$$J = E\big(\eta^2(k) + \rho\mu^2(k)\big) \tag{12.74}$$

The control law that minimizes (12.74) for the system of (12.73) is then given by Theorem 12.4. This control law gives a closed-loop system in which all the zeros of the characteristic equation

$$\tilde{P}(z)\tilde{C}(z) = 0$$

are inside the unit disc. Going back to the original variables results in the characteristic equation

$$P(z)C(z) = \tilde{P}\left(\frac{z}{\bar{r}}\right)\tilde{C}\left(\frac{z}{\bar{r}}\right) = 0$$

All the zeros of this equation are inside the circle $|z| = \bar{r}$.

A simple procedure for obtaining feedback laws that give closed-loop systems with all poles inside the circle $|z| = \bar{r}$ has thus been devised.

Disturbance Reduction

The return difference is

$$H_{rd} = 1 + \mathcal{L} = 1 + \frac{BS}{AR} = \frac{AR + BS}{AR}$$

The inverse of the return difference is a measure of how effectively the closed-loop system eliminates disturbances.

Consider the model of (12.70). Without control the output is

$$y_{ol} = \frac{C}{A}\,e$$

With the LQG-control law, the output becomes

$$y_{lqg} = \frac{R}{P}\,e$$

Elimination of e between these equations gives

$$y_{lqg} = \frac{AR}{PC}\,y_{ol} = \frac{1}{\dfrac{PC}{AR}}\,y_{ol} = \frac{1}{1 + \dfrac{BS}{AR}}\,y_{ol} = \frac{1}{H_{rd}}\,y_{ol} = \mathcal{S}\,y_{ol}$$

The sensitivity function thus tells how much disturbances of different frequencies are attenuated.

Selection of the Sampling Period

There is a substantial difference between the minimum-variance control law discussed in Sec. 12.4 and the LQG-control law discussed in Sec. 12.5 in terms of the influence of the sampling period. The choice of sampling period is critical for the minimum-variance control. A short sampling period gives a high-bandwidth system, which settles quickly. The control actions will also be large when the sampling period is short. In this respect, the minimum-variance control law is similar to the deadbeat control law discussed in Sec. 4.3. The sampling period is less critical for LQG-control. It follows from the analysis of Sec. 11.5 that the control law approaches continuous-time control as the sampling period h goes to zero. The following discussion therefore concentrates on the minimum-variance control law.

Intersample Variation of the Output Variance

The minimum-variance control law minimizes the variance of the output *at the sampling instants*. However, the main objective may be to minimize the continuous-time loss function of (12.7). This may be achieved by first sampling the continuous-time loss function and to minimize the corresponding discrete-time loss function as was discussed in Section 11.1. This results in a complicated design procedure. The minimum-variance control laws are in many cases a sufficiently good approximation. It is useful to investigate the intersample variation of the loss function. This analysis is similar to the analysis of intersample ripple for deterministic systems of Sec. 3.5. An example is used to illustrate the idea.

Example 12.16 Intersample variation of the loss function

Consider the continuous-time system

$$dx = u\,dt + dv \tag{12.75}$$

where $v(t)$ is a Wiener process with incremental covariance $\sigma_v^2\,dt$. Assume that the output is observed without antialiasing filters at times $t_k = k \cdot h$, where h is the sampling period. Hence,

$$y(t_k) = x(k) + \varepsilon(t_k)$$

where $\varepsilon(t_k)$ is a sequence of independent random variables with zero mean and covariance σ_ε^2. Sampling of the system gives

$$x(kh+h) = x(kh) + hu(kh) + v(kh+h) - v(kh)$$
$$y(kh) = x(kh) + \varepsilon(kh)$$

Hence,

$$y(kh+h) = y(kh) + hu(kh) + \varepsilon(kh+h) - \varepsilon(kh) + v(kh+h) - v(kh)$$

The disturbance on the right-hand side may be represented as

$$w(kh+h) = e(kh+h) + ce(kh)$$

where $e(kh)$ is a sequence of independent zero-mean random variables with standard deviation σ.

Simple calculations give

$$c = -1 - \frac{h\sigma_v^2}{2\sigma_\varepsilon^2} + \sqrt{\frac{h\sigma_v^2}{\sigma_\varepsilon^2} + \frac{h^2\sigma_v^4}{4\sigma_\varepsilon^4}}$$
$$\sigma^2 = -\frac{\sigma_\varepsilon^2}{c}$$

The minimum-variance control law for the system is

$$u(kh) = -\frac{1+c}{h}\, y(kh)$$

The standard deviation of the output under minimum-variance control is

$$\mathrm{E}y^2(t) = \sigma^2 \qquad t = h, 2h, \ldots$$

The standard deviation of the state variable x is

$$\mathrm{E}x^2(t) = \sigma^2 - \sigma_\varepsilon^2 \qquad t = h, 2h, \ldots$$

Equation (12.75) is integrated to determine the variance of the state variable between the sampling instants. This gives

$$x(kh+s) = x(kh) + su(kh) + v(kh+s) - v(kh)$$
$$= (1 - \alpha s)x(kh) - \alpha s\varepsilon(kh) + v(kh+s) - v(kh)$$

where

$$\alpha = (1+c)/h$$

We now introduce

$$P_x(s) = \mathrm{E}x^2(kh+s)$$

It then follows that the output variance is

$$P_y(s) = P_x(s) + \sigma_\varepsilon^2 = (1-\alpha s)^2(\sigma^2 - \sigma_\varepsilon^2) + (\alpha s)^2\sigma_\varepsilon^2 + s\sigma_v^2 + \sigma_\varepsilon^2$$

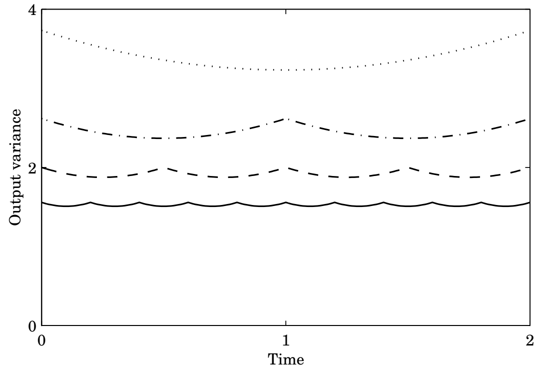

Figure 12.10 Variations of the output variance P_y in Example 12.16 with time for regulators having the sampling periods $h = 0.2$ (solid), $h = 0.5$ (dashed), $h = 1$ (dashed-dotted), and $h = 2$ (dotted).

The function $P_y(s)$ is shown in Fig. 12.10 when $\sigma_\varepsilon = \sigma_v = 1$. Notice that

$$\max_s \bigl(P_y(0) - P_y(s)\bigr) = h^2\sigma_v^2/2$$

The variation in P_y over a sampling interval thus decreases with decreasing h. ■

The analysis is similar in the general case. The only difference is that Theorem 10.5 must be used to compute the state covariance. In the example the variance is largest at the sampling instants. This is not always the case. Also notice that the correct way of dealing with intersample ripple is to sample the continuous-time system and the continuous-time loss functions, as was discussed in Section 11.1.

Computational Aspects

The LQ-control law can be determined by a combination of spectral factorization and solution of linear Diophantine equations. Recall, however, the fundamental difficulty that arises from poor numerical conditioning of polynomial equations (see Sec. 9.6).

12.7 Conclusions

In this chapter optimal-control problems are solved for systems described by input-output models. The results given are limited to single-input–single-output systems. A canonical model for the system, Eq. (12.5), is derived first. This model is characterized by three polynomials, A, B, and C. The underlying continuous-time model may be described as a combination of a time delay and a system with rational transfer functions. The disturbances are characterized as filtered white noise. There are many physical systems that can be described by such models.

Optimal-control problems characterized by quadratic loss functions are solved for the system. A special case where the loss function simply is the variance of the output is considered first. The general problem, in which there is also a penalty on the control variable, is then treated. Both these problems are closely related to the prediction problem for a random process with rational spectral density. This problem is also solved. Practical aspects, such as selection of the sampling period, are also discussed.

The solutions to the optimal-control problems give design tools. The solutions also give insight into the character of the optimal solutions. In particular, they tell that the optimal regulator always cancels stable process zeros that are also zeros for the process disturbances. Stable process poles are canceled only if they are not excited by disturbances. The results also give insight into the relationships between the different design methods. For instance, the LQG-solutions can be interpreted as pole-placement regulators, where the process poles and the observer poles are chosen in special ways.

Calculation of the optimal solution is expressed in terms of two polynomial operations, spectral factorization and solution of Diophantine equations.

12.8 Problems

12.1 Consider the process

$$y(k) = 2\,\frac{q^2 - 1.4q + 0.5}{q^2 - 1.2q + 0.4}\,e(k)$$

where $e(k)$ is white noise with zero mean and unit variance. Determine the optimal m-step-ahead predictor and the variance of the prediction error when $m = 1$, 2, and 3.

12.2 Determine the m-step-ahead predictor for the process

$$y(k) + ay(k-1) = e(k) + ce(k-1)$$

Determine also the variance of the prediction error as a function of m.

12.3 A stochastic process is described by

$$y(k) - 0.9y(k-1) = e(k) + 5e(k-1)$$

(a) Determine an equivalent description such that the zero of a corresponding polynomial C is inside the unit circle. How large is the variance of y?

(b) Determine the two-step-ahead predictor for the process and the variance of the prediction error.

12.4 Assume that the demand for a product in an inventory, $z(k)$, can be described as

$$z(k) = 300 + 10k + y(k)$$

where the time unit is months, and $y(k)$ is described by the process

$$y(k) - 0.7y(k-1) - 0.1y(k-2) = 5e(k)$$

where $e(k)$ is white noise with zero mean and unit variance. Make a prediction and determine the expected standard deviation of the prediction error for August through November when the following data are available:

Month	k	$z(k)$
January	1	320
February	2	320
March	3	325
April	4	330
May	5	350
June	6	370
July	7	375

12.5 Consider the process

$$\begin{aligned} y(k) - y(k-1) + 0.5y(k-2) &= u(k-2) + 0.5u(k-3) \\ &\quad + 0.5\Big(e(k) + 0.8e(k-1) + 0.25e(k-2)\Big) \end{aligned}$$

where e is zero mean white noise with unit standard deviation. Determine the minimum-variance controller and the minimum achievable variance.

12.6 Determine the minimum-variance controller for the system

$$y(k) - 0.5y(k-1) = u(k-2) + e(k) - 0.7e(k-1)$$

where $e(k)$ is white noise with mean 2 and unit variance.

12.7 Consider the process

$$y(k) + ay(k-1) = u(k-2) + e(k) + ce(k-1)$$

(a) Determine the minimum-variance controller.

(b) Discuss the special case $a = 0$.

12.8 Given the system

$$\begin{aligned} y(k) - 1.7y(k-1) + 0.7y(k-2) &= u(k-d) + 0.5u(k-d-1) \\ &\quad + e(k) + 1.5e(k-1) + 0.9e(k-2) \end{aligned}$$

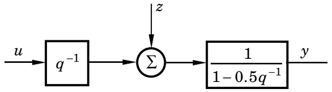

Figure 12.11

(a) Determine the minimum-variance controller and the variance of the output for $d = 1$ and 2.

(b) Simulate the open-loop system and the system controlled with the minimum-variance controller. Compare the output and the control signal for the different cases.

12.9 Consider the process in Fig. 12.11. The disturbance z has the spectral density

$$\phi_z(\omega) = \frac{1}{2\pi} \cdot \frac{1}{1.36 + 1.2\cos\omega}$$

(a) Determine a pulse-transfer function $H(z)$ that gives an output with spectral density ϕ when driven by zero-mean white noise with unit variance.

(b) What is the steady-state variance of y when

$$u(k) = -Ky(k)$$

for $K = 1$?

(c) What is the minimum achievable variance for a proportional controller and how large is the corresponding value of K?

(d) How large is the variance of y when a minimum-variance controller is used?

12.10 Given the system

$$y(k) - 0.25y(k-1) + 0.5y(k-2) = u(k-1) + e(k) + 0.5e(k-1)$$

where $e(k)$ is white noise with unit variance. Assume that the process is controlled with the proportional controller

$$u(k) = -Ky(k)$$

(a) Show that the variance of the output is

$$\frac{2.125 - K}{0.5(1.75-K)(1.25+K)}$$

and that the lowest variance is obtained for $K = 1$, which gives the variance $4/3$.

(b) The expression above is zero for $K = 2.125$. Explain the paradox.

(c) Compute the minimum-variance controller and the resulting output variance.

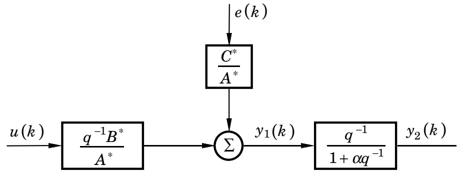

Figure 12.12

12.11 Given the process

$$y(k) - 1.5y(k-1) + 0.7y(k-2) = u(k-2) - 0.5u(k-3) + v(k)$$

(a) Assume that $v(k) = 0$ and compute the deadbeat controller for the system.

(b) Assume that

$$v(k) = e(k) - 0.2e(k-1)$$

where $e(k)$ is white noise. Compute the minimum-variance control law.

(c) What is the steady-state variance of y when the deadbeat and the minimum-variance controllers are used on the system when v is as in b)?

(d) Simulate the system using the different controllers. Study the output and the accumulated loss, that is, the sum of the square of the output.

12.12 Consider the dynamic system

$$y(k) = \frac{B(q)}{A(q)}u(k) + \lambda\frac{C(q)}{D(q)}e(k)$$

where $e(k)$ is white noise and B is stable. The polynomials A, C, and D are assumed to be monic. Determine the minimum-variance controller for the system.

12.13 Use the result from Problem 12.12 to determine the minimum-variance controller for the system

$$y(k) = \frac{bq^{-1}}{1+aq^{-1}}u(k) + (1+cq^{-1})e(k)$$

12.14 Consider the process in Problem 12.13. Assume that the sampling period is doubled; that is, the control signal can be changed only at every second time unit. Determine the minimum-variance controller and compare with the case when the control period is one time unit.

12.15 Consider the system in Fig. 12.12, where e is white noise with zero mean and unit variance. Further,

$$A(q) = q - 0.7 \qquad B(q) = q$$
$$C(q) = 1 - 0.5q \qquad \alpha = -0.8$$

(a) Determine a controller that minimizes the variance of y_1.

(b) Determine the variances of y_1 and y_2 when the controller in (a) is used.

(c) Determine a controller that minimizes the variance of y_2 if only y_2 is measurable, and compute the variances of y_1 and y_2.

(d) Determine a controller that minimizes the variance of y_2 if both y_1 and y_2 are measurable.

(e) What are the variances of y_1 and y_2 when the controller in (d) is used?

12.16 Given the process

$$A(q)y(k) = B(q)u(k) + C(q)e(k) + D(q)v(k)$$

where $v(k)$ is a known disturbance. Determine the minimum-variance controller for the process when $\deg D = \deg B$.

12.17 Determine the LQG-controller given by Theorem 12.4 for the process

$$(1 - 0.9q^{-1})y(k) = u(k-1) + (1 - 0.5q^{-1})e(k)$$

when $\rho = 1$. Calculate the variance of the output and the input for different values of ρ.

12.18 Consider a system with stable inverse. Derive the minimum-variance controller, where the control signal $u(k)$ is allowed to be a function of $y(k-1)$, $y(k-2)$, ..., $u(k-1)$, ... Derive the characteristic equation of the closed-loop system.

12.19 Show that the pulse-transfer function from e to y for (12.5) and (12.55) is given by (12.56). Use (12.45) to derive the minimum-variance controller for a system where

$$A(q) = q^2 - 1.5q + 0.7$$
$$B(q) = q + 0.5$$
$$C(q) = q^2 - q + 0.24$$

Compare with the controller obtained through the identity in (12.17).

12.20 Determine for which systems a digital PID-controller has the same structure as the optimal minimum-variance controller.

12.21 Consider a system described by

$$y(k) = \frac{1}{q-a}\left(bu(k) + \varepsilon(k)\right) + \frac{1}{q-1}w(k)$$

where ε and w are white-noise processes with zero mean and standard deviations σ_ε and σ_w, respectively.

(a) Reduce the system to standard form and determine the minimum-variance controller.

(b) Interpret the controller in (a) as a PI-controller and determine how the gain and the reset time depend on the ratio $\sigma_w^2/\sigma_\varepsilon^2$.

12.22 Consider the minimum-variance control law of (12.31) for a system with an unstable inverse. The output of the closed-loop system is given by

$$y(k) = \frac{F(q)}{q^{d-1}B^{-*}(q)}e(k)$$

Show that the function F/B^{-*} has the series expansion

$$\frac{F(q)}{B^{-*}(q)} = q^{d-1} + f_1 q^{d-2} + \cdots + f_{d-1} + \frac{F_2(q)}{B^{-*}(q)}$$

where $\deg F_2(q) < \deg B^{-*}$ and

$$F_1(q) = q^{d-1} + f_1 q^{d-2} + \cdots + f_{d-1}$$

is the quotient of $q^{d-1}C(q)$ and $A(q)$. Give a convenient way of computing F_2. Use the results of the problem to determine the increase of the minimum-variance due to unstable system zeros.

12.23 Determine the intersample ripple of the loss function when the process

$$dx_1 = x_2\,dt$$
$$dx_2 = u\,dt + dv$$
$$y(t_k) = x_1(t_k) + \varepsilon(t_k)$$

is controlled by the minimum-variance regulator. The process $v(t)$ is a Wiener process with incremental covariance $\sigma_v^2\,dt$, and $\varepsilon(t_k)$ is white measurement noise with zero mean and variance σ_ε^2.

12.24 Consider the process in Example 12.16. Determine the control law with sampling period h that minimizes

$$\lim_{T\to\infty} \mathrm{E}\,\frac{1}{T}\int_0^T x^2(s)\,ds$$

and compare it with the minimum-variance control.

12.25 Consider a process subject to a disturbance that is characterized as a Wiener process with incremental covariance dt. Determine the prediction error of the minimum-variance in each case. Use different prediction horizons and sampling periods.

(a) The process has an unstable zero $z = b > 1$.

(b) The process has an unstable pole $z = a > 1$.

12.26 Consider the system in Problem 12.23 with an extra time delay of 1 s. Determine the minimum-variance as a function of the sampling period.

12.27 Consider the system in Problem 12.23. Determine the output variance as a function of the input covariance for different sampling periods.

12.28 Consider the system

$$y(k) = \frac{1}{q-0.999}u(k) + \frac{q}{q-0.7}e(k)$$

Determine the minimum-variance control law for the system. Compare it with a proportional feedback that gives a corresponding response rate. Discuss the relative merits of the control laws by calculating their loop gains and return differences. Explain why the minimum-variance control is inferior. (*Hint:* A bad optimization problem gives a bad optimal regulator.)

12.29 Given the system

$$y(k) = 1.4y(k-1) - 0.65y(k-2) + u(k-1) - 0.2u(k-2) + e(k) + 0.4e(k-1)$$

where $e \in N(0,2)$

(a) Determine the minimum-variance controller.

(b) Determine the deadbeat controller.

(c) Compute the variance of y when the controllers in (a) and (b), respectively, are used.

12.30 Consider the system

$$y(k) + ay(k-1) = u(k-1) + e(k) + ce(k-1)$$

where $e \in N(0,1)$. We want to determine the minimum-variance controller for the process but the value of c is unknown.

(a) Assume in the design that $c = 0$ and determine the minimum-variance controller for the system

$$y(k) + ay(k-1) = u(k-1) + e(k)$$

How large will the output variance be if this controller is used on the true system?

(b) Assume instead that $c = \hat{c}$ and redo the calculations in (a).

12.31 Consider the stochastic process

$$y(k+2) - 1.1y(k+1) + 0.3y(k) = e(k+2) - 1.25e(k+1)$$

where $e \in N(0,1)$.

(a) Determine the two-step-ahead predictor for $y(k)$.

(b) Calculate the variance of the prediction error.

12.32 Given the system

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

where

$$A(q) = q^3 - 1.7q^2 + 0.8q - 0.1$$
$$B(q) = 2(q - 0.9)$$
$$C(q) = q^2(q - 0.1)$$

and $e(k) \in N(0,1)$.

(a) Determine the minimum-variance controller for the system.

(b) Determine the variance of the output when controlling the system with the controller in (a).

(c) Redo the calculations in (a) and (b) when

$$B(q) = 2(0.9q - 1)$$

12.33 Consider the process in Example 12.9. Compute the output variance when the controller does not cancel the zero, that is, when the controller is obtained from the identity

$$zC = AR + BS$$

Compare the variances.

12.34 Consider the process in Example 12.9. Compute the controller that minimizes the loss function (12.7).

12.35 Show that a system with the input-output description

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

where

$$A(q) = q^n + a_1 q^{n-1} + \cdots + a_n$$
$$B(q) = b_1 q^{n-1} + \cdots + b_n$$
$$C(q) = q^n + c_1 q^{n-1} + \cdots + c_n$$

has the following state-space description

$$x(k+1) = \Phi x(k) + \Gamma u(k) + Ke(k+1)$$
$$y(k) = Cx(k)$$

where the state vector has dimension $n+1$ and

$$\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ 0 \end{pmatrix} \quad K = \begin{pmatrix} 1 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

12.36 Consider the system in Problem 12.35. Assume that the polynomial $C(z)$ has all its zeros inside the unit disc. Show that the Kalman filter for the system can be written as

$$\hat{x}(k+1 \mid k) = \Phi\hat{x}(k \mid k) + \Gamma u(k)$$
$$\hat{x}(k+1 \mid k+1) = \hat{x}(k+1 \mid k) + K\Big(y(k+1) - C\hat{x}(k+1 \mid k)\Big)$$

and that the characteristic polynomial of the filter is $zC(z)$.

12.37 Consider the system in Problem 12.35. Assume that minimization of a quadratic loss function gives the feedback law

$$u(k) = -L\hat{x}(k \mid k)$$

Show that the controller has the pulse-transfer function

$$H_c(z) = zL\Big(zI - (I - KC)(\Phi - \Gamma L)\Big)^{-1}\Gamma$$

Show that the results are the same as those given by

$$\begin{aligned} H_c(z) &= L_v(\Phi - KC)\Big(zI - (I - \Gamma L_v)(\Phi - KC)\Big)^{-1}(I - \Gamma L_v)K + L_v K \\ &= zL_v\Big(zI - (\Phi - KC)(I - \Gamma L_v)\Big)^{-1} K \end{aligned} \tag{12.76}$$

12.38 Consider the system in Problem 12.35. Assume that $b_1 \neq 0$. Determine the minimum-variance strategy using the state-space representations in (12.38) and in Problem 12.37. Compare the results. (*Hint:* The minimum-variance control corresponds to $L = [-a_1 \; 1 \; 0 \; \cdots \; 0]$.)

12.39 Derive the expressions for the transfer function $H_c(z)$ in Eq. (12.76) using the matrix inversion Lemma B.1 in Appendix B.

12.40 Show that the transfer function $H_c(z)$ in Eq. (12.76) can be written as

$$H_c(z) = \frac{S(z)}{R(z)} = \alpha + (L - \alpha C)(zI - \Phi + \Gamma L + KC - \alpha \Gamma C)^{-1}(K - \Gamma \alpha)$$

where $\alpha = L_v K$. Show that this expression is equivalent to

$$\frac{S(z)}{R(z)} = \frac{S_0(z) + \alpha A(z)}{R_0(z) - \alpha B(z)}$$

where $S_0(z)$ and $R_0(z)$ is the solution to the Diophantine equation

$$A(z)R(z) + B(z)S(z) = P(z)C(z)$$

with $\deg R(z) = n$ and $\deg S(z) < n$.

12.9 Notes and References

The treatment of the linear quadratic case is in the spirit of Wiener's work; see Wiener (1949), Newton, Gould, and Kaiser (1957), and Youla, Bongiorno, and Jabr (1976).

A thorough discussion of prediction and minimum-variance control is found in Åström (1970), which is based on Åström (1965, 1967). A similar approach to the stochastic-control problem is found in Box and Jenkins (1970). The theorem for minimum-variance control of systems with unstable inverses was first published in Peterka (1972). An algebraic approach to the multivariable LQ- and minimum-variance control problems is given in Kučera (1979). Also see Kučera (1984, 1991), and Mosca, Giarre, and Casavola (1990). Choice of sampling interval for stochastic control is discussed in the books mentioned before, and also in MacGregor (1976).

The intersample variation of the variance is discussed in De Souza and Goodwin (1984) and Lennartson and Söderström (1986).

13

Identification

13.1 Introduction

The notion of a *mathematical model* is fundamental to science and engineering. A model is a very useful and compact way to summarize the knowledge about a process. A model is also a very effective tool for education and communication. The design methods in the previous chapters assume that models for the process and the disturbances are given. The process models can sometimes be obtained from first principles of physics. It is more difficult to get the models of the disturbances, which are equally important. These models often have to be obtained from experiments. The types of models that are needed for the design methods presented here are either state-space models (internal models) or input-output models (external models). The models for the disturbances are for the internal models given as dynamic systems driven by white noise. For external models the disturbances are given in terms of spectral densities and covariance functions. Models for disturbances can, however, only rarely be determined from first principles. Experiments are thus often the only way to get models for the disturbances.

A process cannot be characterized by *one* mathematical model. A process should be represented by a *hierarchy* of models ranging from detailed and complex simulation models to very simple models, which are easy to manipulate analytically. The simple models are used for exploratory purposes and to obtain the gross features of the system behavior. The complicated models are used for a detailed check of the performance of the control system. The complicated models take a long time to develop. Between the two extremes, there may be many different types of models. The trademark of good engineering is to choose the right model for each specific purpose.