Nyquist Noise and Pink Noise

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Harry Nyquist 1889 - 1976

Contributions
- Johnson-Nyquist noise
- Nyquist frequency

Career
- Nilsby Värmland
- Folkskola 6 year Nilsby
- Teachers college USA
- BS EE University North Dakota
- PhD Physics Yale 1917
- Bell Labs 1917-1954
- IEEE Medal of Honor 1960
- ASME Nyquist lecture

Fig. 10-1. H. Nyquist, who originated many fundamental concepts of communications, including the criterion for stability of negative feedback amplifiers and sampling theory as applied to digital systems.

Statistical Physics

- The galvanometer paradox
- The fluctuation-dissipation theorem; fluctuations are associated with energy dissipation
- Einstein (Brownian motion), Johnson-Nyquist (Resistor noise) and many other
- Boltzmann's Equipartition Principle

  Consider a collection of particles in thermal equilibrium, all particles then have the average energy $\frac{1}{2}k_BT$, where $k_B = 1.38 \times 10^{-23}$ [J/Kelvin] is Boltzmann's constant

- For a system in thermal equilibrium energy is distributed so that each state (degree of freedom) has the energy $\frac{1}{2}k_BT$

The Problem of Units 2 and π

Notice different units Hz or rad/s, positive or negative frequencies and different ways of placing the factor 2π

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} r(t)dt, \qquad r(t) = \int_{-\infty}^{\infty} e^{i\omega t}\phi(\omega)d\omega$$

Using f in Hz instead of ω in radians/s, the relations between spectral density and covariance function becomes

$$\bar{S}(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} r(t)dt = 2\int_{0}^{\infty} r(t)\cos(2\pi f t)dt$$
$$r(t) = \int_{-\infty}^{\infty} e^{2\pi i f t}\bar{S}(f)df = 2\int_{0}^{\infty} \bar{S}(f)\cos(2\pi f t)df$$

where $\bar{S}(f) = 2\pi\phi(2\pi f)$. A good rule is to define spectral density so that the area under the spectral density represents the mean square fluctuations.

Johnson-Nyquist Noise

A classic paper on noise in electric amplifiers was written by Schottky in 1918 who conjectured two physical mechanism, shot noise and thermal noise. Johnson at AT&T made very careful measurements of thermal noise in 1928. He found that the thermal noise in a resistor was proportional to resistance R and temperature T. His colleague Nyquist gave a very nice physical explanation by combining statistical thermodynamics with transmission line theory. In particular Nyquist found that the mean square voltage fluctuations when current flows through a resistor is

$$V^2 = 4k_BTR\Delta f$$

where $k_B = 1.38 \times 10^{-23}$ [J/Kelvin] is Boltzmann's constant, T temperature and f [Hz] the bandwidth.

A Beautiful Example of Stochastic Modeling

Johnson: *The results were discussed with Dr. H. Nyquist, who in a matter of a month or so came up with the famous formula*

$$V^2 = 4k_BTR\Delta f.$$

for the effect, based essentially on the thermodynamics of a telephone line, and covering almost all one need to know about thermal noise.

Theoretical physicist from Bell: *Nyquist's fusing of concepts from two quite different fields, statistical mechanics and electrical engineering, points out what has been a particular strength of Bell Labs work in theoretical physics: the diversity of expertise among the theoretical staff, and the propensity of many of them to shift their attention from one area to another, transferring useful concepts in the process.*

Nyquist's Derivation

E. B. Johnson Thermal agitation of electricity in conductors. Phys. Rev. 32(1928)97-109.

H. Nyquist Thermal agitation of electric charge in conductors. Phys. Rev. 32(1928)110-113.

Consider a long non-dissipative transmission line with a resistor R at each end. Let the inductance and capacitance per unit length be L and C and choose $R = \sqrt{(L/C)}$ which gives no reflections. If the resistors are at the same temperature and in thermal equilibrium. Energy is thus transmitted between the resistors. The transmitted energy can be trapped by short circuiting the line which gives standing waves with perfect reflection. By computing the energy stored in the line we can obtain the energy transmitted by the resistors.

Nyquist's Derivation

To compute the trapped energy we introduce the length of the line ℓ and the transmission velocity v. The standing wave has frequencies $f = nv/2\ell$, where n is an integer.. Provided that ℓ is sufficiently large the frequencies can be arbitrarily dense. The number of modes in a frequency interval Δf is thus $n = 2\ell\Delta f/v$. According to Boltzmann's equipartition law the energy of the standing waves is then $kTn = 2\ell k_BT\Delta f/v$. Since the time to pass the cable is ℓ/v the average power is thus $2k_BT\Delta f$. The power transmitted from each resistor is thus $2k_BT\Delta f$. Let the average voltage generated by thermal noise in a resistor be V, the current is then $I = V/2R$ and the power is $RI^2/4$ equating this with the transmitted power gives

$$V^2 = 4k_BTR\Delta f$$

Single sided spectrum!

Relations to SDE

Since formally $V = dw/dt$ the voltage variations has the covariance function

$$r_V(t) = 2k_B TR\delta(t)$$

The corresponding spectral density is

$$\phi(\omega) = \frac{2k_B TR}{2\pi} = \frac{k_B TR}{\pi}\,[V^2 s/rad] = 2k_B TR\,[V^2/Hz]$$

Thermal noise in a resistor can thus be represented by white noise with the spectral density $k_B RT/\pi[V^2\, s/rad] = 2k_B TR\,[V^2/Hz]$. The corresponding Wiener process has incremental covariance $Edw^2 = 2Rk_b Tdt$. *Notice that the incremental covariance is proportional to the resistance R and temperature T.*

Nyquist's formula is $4k_B TR$ since he only considers positive frequencies his bandwidth half of ours.

Sources of Noise in Electro-Mechanical Systems

Spectrum and covariance functions:

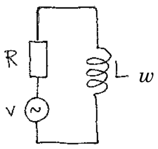
$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} r(t)dt, \qquad r(t) = \int_{-\infty}^{\infty} e^{i\omega t}\phi(\omega)d\omega$$

- Johnson-Nyquist resistor noise is modeled as a white noise voltage in series with the resistor. The voltage is derivative of Wiener process with incremental covariance $r_w dt = 2k_B TRdt$, where $k_B = 1.38 \times 10^{-23}$ [J/Kelvin] is Boltzmann's constant, R is the resistance and T the absolute temperature of the resistor.
- Thermal noise in mechanical systems modeled as white noise force, derivative of Wiener process with incremental covariance $r_f dt = 2k_B TCdt$ in series with the damper, C is the damping constant [Ns/m].

Johnson-Nyquist Noise

Consider an RL circuit which is described by

$$L\frac{dI}{dt} + RI = V$$



Assuming that voltage variations are modeled as white noise, i.e. $V = dw/dt$, where is a Wiener process with incremental covariance $r_w dt$. The system can then be described by the stochastic differential equation

$$LdI + RIdt = dw, \qquad dI = -\frac{R}{L}Idt + \frac{1}{L}dw$$

Johnson-Nyquist Noise ...

Consider the stochastic differential equation

$$dI = -\frac{R}{L}Idt + \frac{1}{L}dw$$

Introduce the variance of the current fluctuations $P = EI^2$

$$\frac{dP}{dt} = -2\frac{R}{L}P + \frac{r_w}{L^2}$$

In steady state we have $r_w = 2RLP$. The average energy stored in the inductor is $LEI^2/2 = PL/2$. Boltzmann's equipartition law gives

$$\frac{1}{2}LP = \frac{1}{2}k_B T$$

which gives $PL = k_B T$ and $r_w = 2k_B TR$.

In General

It is straight forward to analyse linear stochastic differential equations (SDE).

- Formulate the stochastic differential equation
- When the model is given the evolution of covariances are given by linear differential equations
- For simple problems we can solve the equations analytically
- Steady state variances are given by linear equations
- For systems of higher order we can use numeric linear algebra, Matlab programs are available

Thermal Noise in an Accelerometer

A simple model of a MEMS accelerometer is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

Collisions of air molecules in thermal motion generates the forces f which we model as white noise. To analyze the system we first write it in state space form by introducing $x_1 = x$ and $x_2 = dx/dt$. The equation then becomes

$$\begin{pmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}f\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{m}\end{pmatrix} f$$

The model can be written in standard form as

$$\frac{dx}{dt} = \begin{pmatrix}0 & 1\\ -k/m & -c/m\end{pmatrix}x + \begin{pmatrix}0\\ 1/m\end{pmatrix} f$$

Thermal Noise in an Accelerometer ...

Assuming that the force is white noise the system is described by the stochastic differential equation

$$dx = \begin{pmatrix}0 & 1\\ -k/m & -c/m\end{pmatrix}xdt + \begin{pmatrix}0\\ 1/m\end{pmatrix} df$$

To determine the magnitude of the fluctuations we use the variance equation

$$\begin{aligned}\frac{dP}{dt} &= AP + PA^T + R\\ &= \begin{pmatrix}0 & 1\\ -k/m & -c/m\end{pmatrix}P + P\begin{pmatrix}0 & 1\\ -k/m & -c/m\end{pmatrix}^T + \begin{pmatrix}0 & 0\\ 0 & r_f/m^2\end{pmatrix}\end{aligned}$$

The steady state solution is given by

$$AP + PA^T + R = 0$$

Steady State Variations

We have

$$\begin{aligned}AP &= \begin{pmatrix}0 & 1\\ -k/m & -c/m\end{pmatrix}\begin{pmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{pmatrix}\\ &= \begin{pmatrix}p_{12} & p_{22}\\ -(kp_{11}+cp_{12})/m & -(kp_{12}+cp_{22})/m\end{pmatrix}\end{aligned}$$

The equation $AP + PA^T + R = 0$ then becomes

$$\begin{aligned}2p_{12} &= 0\\ p_{22} - \frac{kp_{11}+cp_{12}}{m} &= 0\\ -2\frac{kp_{12}+cp_{22}}{m} + \frac{r_f}{m^2} &= 0\end{aligned}$$

Hence $p_{12} = 0$, $p_{22} = \dfrac{r_f}{2cm}$ and $p_{11} = \dfrac{r_f}{2ck}$

Steady State Variations ...

We have $p_{11} = \dfrac{r_f}{2ck}$, $p_{12} = 0$ and $p_{22} = \dfrac{r_f}{2cm}$. To determine r_f we use Boltzmann's equipartition law. This gives

$$\frac{1}{2}kEx_1^2 = \frac{1}{2}kp_{11} = \frac{1}{2}\frac{kr_f}{2ck} = \frac{1}{2}c_BT$$

This gives $r_f = 2ck_BT$. Notice that we get the same result if compute the average energy based on the velocity, i.e.

$$\frac{1}{2}mEx_2^2 = \frac{1}{2}mp_{22} = \frac{1}{2}\frac{mr_f}{2cm} = \frac{1}{2}k_BT$$

We thus find that thermal noise gives the following variances of position and velocity of the mass

$$Ex_1^2 = p_{11} = \frac{r_f}{2ck} = \frac{k_BT}{k}, \qquad Ex_2^2 = p_{22} = \frac{r_f}{2cm} = \frac{k_BT}{m}$$

Thermal Fluctuations

The effect of thermal fluctuations on the accelerometer can be represented as a force which is white noise with covariance function

$$r_f(t) = 2ck_BT\delta(t)$$

The corresponding Wiener process has incremental covariance $2ck_BTdt$. The corresponding spectral density is

$$\phi(\omega) = \frac{2ck_BT}{2\pi} = \frac{k_BcT}{\pi}\ [N^2/rad/s] = 2k_BcT\ [N^2/Hz]$$

Notice that spectral density is proportional to the damping coefficient c.

The fluctuation-dissipation theorem!!

Pink Noise

Resistors, semiconductors and bonds all have many small defects that create variations in conductivity. The conductivity variations generate signal variations when driven by a current. The fluctuations depend on the strength of the current. The noise is not white but it varies as $1/f$ and appears as a slow drift. There are many names The noise goes by many names *pink noise*, *flicker noise* or $1/f$ *noise*. The noise level is often specified by giving the frequency f_0, where the $1/f$ noise matches the level of input white noise or current noise. The level varies significantly with the type and quality of the operational amplifier, typical ranges are $f_0 = 1 - 1000$ Hz.

Pink Noise Behavior

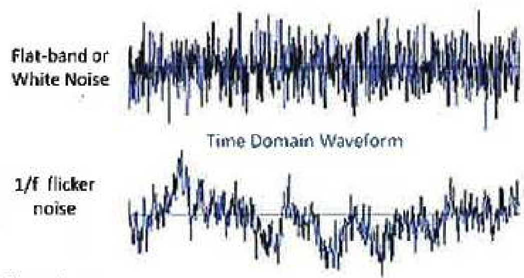


Figure 1.

Modeling Pink Noise

Ensemble of first order system in thermal equilibrium. Equipartition law implies that they have the same energy. Let the bandwidth of a subsystem be a, spectral density of an individual system is proportional to

$$\phi = \frac{a}{\omega^2 + a^2},$$

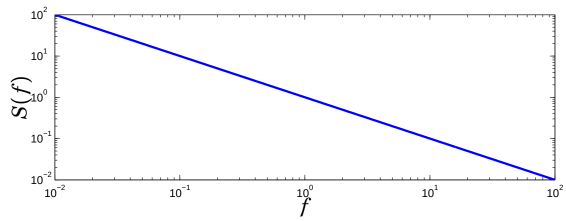
Assuming all with energy logarithmically distributed, then

$$\int_0^\infty \frac{a}{a^2+\omega^2}d\log a = \int_0^\infty \frac{1}{a^2+\omega^2}da = \frac{1}{\omega}\arctan\frac{a}{\omega}\Big|_0^\infty = \frac{\pi}{2\omega}.$$

The spectral density of ensemble is thus proportional to $1/\omega$ or $1/f$

$$\int_{f_1}^{f_2}\frac{df}{f} = \log\frac{f_2}{f_1}$$

Pink Noise



$$S(f) = \frac{1}{f}, \qquad \int_{f_1}^{f_2} S(f) = \log\frac{f_2}{f_1}$$

The same energy for each octave

Pink Noise

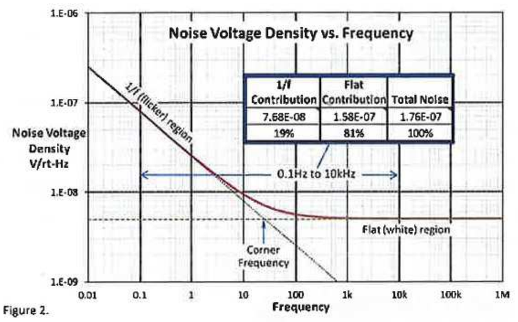


Figure 2.

Sources of Noise in Electrical Circuits

- Shot noise
  - White noise with incremental covariance $r_s dt = qI\,dt$ where $q = 1.6\times 10^{-19}\ C$ is the charge of the electron and I is the current. Typical op-amps $0.1fA/\sqrt{Hz}$ to $10pA/\sqrt{Hz}$
- Johnson-Nyquist noise $4k_BTR\Delta B$
  - White with incremental covariance $r_w dt = 2k_BTRdt$
- Amplifier noise
  - Voltage noise OP27: $3nV/\sqrt{Hz}$
  - Current noise shot noise in bias current
    OP27: $1pA/\sqrt{Hz}$
    AD795: $0.6fA/\sqrt{Hz}$
  - $1/f$ noise often specified as corner frequency typical values for op-amps 2Hz to 2kHz