Exercise 1

1. Reading assignment
   - Chapter 2 in the course book (refresh in mind).
   - Read § 4.1 and § 4.2 in the course book.
2. Problem 2.2 in the course book
3. Problem 2.4 in the course book
4. † Problem 2.5 in the course book[1]
5. Consider the following Hermitian block matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix},$$

   with $\Phi_{11}' = \Phi_{11}$, $\Phi_{22}' = \Phi_{22}$ and $\Phi_{12}' = \Phi_{21}$. Prove that $\Phi > 0$ only if $\Phi_{11} > 0$ and $\Phi_{22} > 0$.
6. Consider a space of continuous functions with continuous derivatives. Which of the following expressions qualifies as a norm?
   (a) $\sup_t |\dot{u}(t)|$
   (b) $|u(0)| + \sup_t |\dot{u}(t)|$
7. For the space of the functions $f : \mathbb{R} \to \mathbb{R}^n$, prove that the definition

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \operatorname{trace}\left(g'(t) f(t)\right) dt$$

   qualifies as an internal product.
8. Consider a space of continuous functions on $f : [0,1] \to \mathbb{R}$ with standard inner product and norm

$$\langle f, g \rangle := \int_0^1 g(t) f(t) dt, \quad ||f|| := \sqrt{\int_0^1 f(t)^2 dt}.$$

   Consider a third order polynomial $v = x^3$ and a subspace $\mathcal{S}$ spanned by $u_1 = x$ and $u_2 = x^2$. Prove that

$$\operatorname*{argmin}_{u \in \mathcal{S}} ||v - u|| = \frac{4}{3} x^2 - \frac{2}{5} x.$$

   (Use the projection theorem.)
9. † Calculate the $\mathcal{H}^2$ and $\mathcal{H}^\infty$ distances between

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{1}{s+1} e^{-\theta s}$$

   for $\theta = \{0.01, 0.1, 1\}$. How do the distances depend on θ?

[1] Problems marked with † are the "hand-in" assignments.