H_∞ loop shaping

Robust Control Course

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Intro

Classical loop shaping

- frequency domain method for controller design
- closed-loop design objectives are expressed in terms of open-loop transfer functions
- open-loop transfer functions are shaped using lead/lag compensators etc.
- Stability and robustness issues are handled using Nyquist criteria

Our aim is to generalize these ideas for MIMO problems

Robust stability framework from Lecture 5 will play an important role in this generalization

Single loop setting

Consider a single loop setting

As before we denote

$$L_o = PC, \quad S_o = (1 + L_o)^{-1}, \quad T_o = I - S_o,$$
$$L_i = CP, \quad S_i = (1 + L_i)^{-1}, \quad T_i = I - S_i$$

and recall that

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} S_o & -S_oP & -S_o & T_o \\ S_iC & -T_i & -S_iC & -S_iC \end{bmatrix} \begin{bmatrix} r \\ d_i \\ d_o \\ n \end{bmatrix}$$

Requirements on the closed-loop transfer functions

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} S_o & -S_oP & -S_o & T_o \\ S_iC & -T_i & -S_iC & -S_iC \end{bmatrix} \begin{bmatrix} r \\ d_i \\ d_o \\ n \end{bmatrix}$$

S_o, S_oP - small at low frequencies for tracking and disturbance rejection

T_o - small at high frequencies for noise rejection

T_o, S_iC - small with roll-off at high frequencies for robust stability subject to additive and multiplicative uncertainties

S_iC, T_i - not too large to prevent large control effort

This is a rational behind mixed sensitivity problem

Sometimes mixed sensitivity framework is not transparent

We translate these requirements to the open-loop as in classical loop shaping

Translating requirements to the open-loop transfer functions

If $\underline{\sigma}(L_o) >> 1$ (the loop gain is high), then

$$\bar{\sigma}(S_o) \approx \frac{1}{\underline{\sigma}(L_o)} \quad \text{and} \quad \bar{\sigma}(S_o P) \approx \frac{1}{\underline{\sigma}(C)}$$

If $\bar{\sigma}(L_o) << 1$ (the loop gain is low), then

$$\bar{\sigma}(T_o) \approx \bar{\sigma}(L_o) \quad \text{and} \quad \bar{\sigma}(CS_i) = \bar{\sigma}(S_o C) \approx \bar{\sigma}(C)$$

For tracking and disturbance rejection we need

$\underline{\sigma}(L_o) >> 1$ and $\underline{\sigma}(C) >> 1$ at low frequencies

For noise rejection and robust stability (for additive uncertainty)

$\bar{\sigma}(L_o) << 1$ and $\bar{\sigma}(C) << 1$ at high frequencies

Sometimes also L_i should be taken into account.

What should we do with the stability requirement?

- in SISO case it can be captured by Nyquist criteria
- however, this is not readily extendable for the MIMO case

The main idea of the proposed approach is to capture stability issues via robust stability framework from Lecture 5:

- being far from critical point $\Leftrightarrow$ increasing stability margins
- can be handled via H_∞ optimization framework

The idea of H_∞ loop shaping

- Design the weights W_i and W_o to shape $P_s = W_o P W_i$,
  which represents the open loop

- For the shaped plant P_s synthesize the controller K_s, maximizing stability margin towards unstructured uncertainty (H_∞ optimization)

- Construct the controller $K = W_i K_s W_o$

What is suspicious in this algorithm?

The idea of H_∞ loop shaping (contd.)

- It is suspicious that we shape $P_s = W_o P W_i$,
  while the real open-loop transfer matrix is $L_o = P W_i K W_o$

- The shape of L_o depends on K, synthesized via the robust stabilization procedure, and can be different from the shape of P_s

- But there exists a specific uncertainty model which guarantees only a mild deterioration in the shape of L_o ...

Our main technical step will be to find a specific uncertainty model for robust stabilization procedure, which

- results in a simple H_∞ optimization problem
  having an easy to compute solution

- guarantees that the shapes of P_s and L_o are similar

Robust stability subject to lcf uncertainty

Lcf uncertainty model $P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$

with $\|\tilde{\Delta}\|_\infty = \|[\tilde{\Delta}_N \ \tilde{\Delta}_M]\|_\infty \leq 1/\gamma = \alpha$

The corresponding generalized plant is

$$G = \left[\begin{array}{c:c} 0 & I \\ \tilde{M}^{-1} & -P \\ \hdashline \tilde{M}^{-1} & -P \end{array}\right]$$

Robust stability is equivalent to

$$\|\mathcal{F}_l(G,K)\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma.$$

Consider minimal realization of the plant

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

(For simplicity, we assume that the plant is strictly proper)

Using the material from Lecture 2, it is easy to verify that

$$[\tilde{N}\ \tilde{M}] = \left[\begin{array}{c|cc} A+LC & B & L \\ \hline C & 0 & I \end{array}\right], \quad G = \left[\begin{array}{c|c:c} A & -L & B \\ \hline 0 & 0 & I \\ C & I & 0 \\ \hdashline C & I & 0 \end{array}\right].$$

Remark: Since $D_{11}' D_{11} = I$, we have that

$$\gamma_{opt} > \|I\| = 1 \quad \Rightarrow \quad \alpha_{opt} = 1/\gamma_{opt} < 1.$$

α_{opt} will be referred to as "maximal stability radius".

At this point we can apply solution of the standard H_∞ problem. Two Hamiltonian matrices are

$$H = \begin{bmatrix} A - \frac{1}{\gamma^2-1}LC & \frac{1}{\gamma^2-1}LL^* - BB^* \\ -\frac{\gamma^2}{\gamma^2-1}C^*C & -(A - \frac{1}{\gamma^2-1}LC)^* \end{bmatrix},$$

$$J = \begin{bmatrix} (A+LC)^* & -C^*C \\ 0 & -(A+LC) \end{bmatrix}.$$

Note that in this special case $Y = 0$ (solution of ARE associated with J).

Theorem

Assuming $D = 0$, there exists a stabilizing controller K such that

$$\|\mathcal{F}_l(G, K)\|_\infty < \gamma$$

if and only if $\gamma > 1$, $H \in \mathrm{dom}(\mathrm{Ric})$ and $X = \mathrm{Ric}(H) \geq 0$.

Remark: The result depends on the choice of L, i.e., on the choice of coprime factors.

Normalized coprime factorization

We choose left coprime factorization to be "normalized", namely, to satisfy

$$\tilde{N}(s)\tilde{N}^{\sim}(s) + \tilde{M}(s)\tilde{M}^{\sim}(s) = I$$

or equivalently

$$\begin{bmatrix} \tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} \tilde{N}^{\sim}(s) \\ \tilde{M}^{\sim}(s) \end{bmatrix} = I$$

To construct normalized lcf we need to choose $L = -YC^*$, where Y is the stabilizing solution of

$$AY + YA^* - YC^*CY + BB^* = 0$$

Proof: ...

Robust stability subject to normalized lcf uncertainty

Since $\tilde{M}$, $\tilde{N}$ is the normalized lcf, multiplication by $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ does not change the norm. Therefore,

$$\begin{aligned}
\|\mathcal{F}_l(G,K)\|_\infty &= \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1} \tilde{M}^{-1} \right\| \\
&= \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} P & I \end{bmatrix} \right\| = \left\| \begin{bmatrix} T_i & KS_o \\ S_oP & S_o \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} K & I \end{bmatrix} \right\| = \left\| \begin{bmatrix} T_o & PS_i \\ KS_i & S_i \end{bmatrix} \right\|
\end{aligned}$$

Does not depend on factorization

All closed-loop transfer function are equally penalized

- well balanced optimization
- does not tend to perform undesirable pole-zero cancellations

Explicit expression for γ_{opt}

Once normalized lcf is used, explicit expression for α_{opt} can be derived.

Theorem

The maximal stability radius for the robust stability problem with a normalized lcf uncertainty is given by

$$\alpha_{opt} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \right\|_H^2} < 1$$

where $\| \cdot \|_H$ stands for the Hankel norm.

Remark: Hankel norm is the maximal Hankel singular value, namely,

$$\left\| \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right\|_H = \sqrt{\rho(PQ)},$$

where P and Q are the controllability and observability Gramians, respectively.

State-space formulae for the solution

The state-space solution can be derived in terms of the following equations

$$AY + YA^* - YC^*CY + BB^* = 0 \quad (1)$$

$$Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0 \quad (2)$$

Theorem

Assume $D = 0$ and denote $L = -YC^*$, where $Y \geq 0$ is the stabilizing solution of (1). Let Q be a solution of (2). Given $\gamma > 0$, there exists stabilizing controller satisfying $\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_\infty < \gamma$ iff

$$\gamma > \gamma_{opt} = \left(1 - \|\tilde{N} \ \tilde{M}\|_H^2\right)^{-1/2} = (1 - \rho(YQ))^{-1/2}$$

Moreover, a controller achieving $\gamma > \gamma_{opt}$ can be given by

$$K(s) = \left[\begin{array}{c|c} A - BB^*X - YC^*C & -YC^* \\ \hline -B^*X & 0 \end{array}\right], \ X = \frac{\gamma^2}{\gamma^2 - 1} Q \left(I - \frac{\gamma^2}{\gamma^2 - 1} YQ\right)^{-1}.$$

Relation to Gain and Phase Margins

It turns out that in the SISO case the stability radius $\alpha = \|\mathcal{F}_l(G,K)\|_\infty^{-1}$ can be related to the classical stability margins.

Theorem

Let P be a SISO plant and K be a stabilizing controller. Then

$$\begin{aligned} \text{gain margin} &\geq \frac{1+\alpha}{1-\alpha}, \\ \text{phase margin} &\geq 2\arcsin(\alpha). \end{aligned}$$

Proof: For SISO system at every ω

$$\begin{aligned} \alpha &= \frac{1}{\|\ldots\|_\infty} \leq \frac{|1+P(j\omega)K(j\omega)|}{\left\| \begin{bmatrix} 1 \\ K \end{bmatrix} \begin{bmatrix} 1 & P \end{bmatrix} \right\|} = \\ &= \frac{|1+P(j\omega)K(j\omega)|}{\left\| \begin{bmatrix} 1 \\ K \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 & P \end{bmatrix} \right\|} = \frac{|1+P(j\omega)K(j\omega)|}{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K(j\omega)|^2}}. \end{aligned}$$

So at frequencies where $k := -PK \in R^+$ we have

$$\begin{aligned} \alpha &\leq \frac{|1-k|}{\sqrt{(1+|P|^2)(1+k^2/|P|^2)}} \leq \\ &\leq \frac{|1-k|}{\sqrt{\min_P\{(1+|P|^2)(1+k^2/|P|^2)\}}} = \frac{|1-k|}{|1+k|} \end{aligned}$$

from which the gain margin result follows.

Similarly at frequencies where $PK = -e^{\theta}$

$$\begin{aligned} \alpha &\leq \frac{|1-e^{\theta}|}{\sqrt{(1+|P|^2)(1+1/|P|^2)}} \leq \\ &\leq \frac{|1-e^{\theta}|}{\sqrt{\min_P\{(1+|P|^2)(1+1/|P|^2)\}}} = \\ &= \frac{2|\sin(\theta/2)|}{2} \end{aligned}$$

which implies the phase margin result.

Intermediate summary

We saw that the problem of robust stability subject to normalized lcf uncertainty has many appealing properties:

- maximization of stability radius results in well balanced optimization
- admits explicit solution (no iterations needed)
- related with classical stability margins

This is the time to go back to our original problem:

Are there guarantees that the shapes of P_s and L_o are similar?

Degradation at the low frequencies

At the low frequencies we need large $\underline{\sigma}(L_o)$ and $\underline{\sigma}(L_i)$. It is easy to verify that

$$\underline{\sigma}(L_o) = \underline{\sigma}(PW_iK_sW_o) = \underline{\sigma}(W_o^{-1}P_sK_sW_o) \geq \underline{\sigma}(P_s)\underline{\sigma}(K_s)\frac{1}{\kappa(W_o)},$$
$$\underline{\sigma}(L_i) = \underline{\sigma}(W_iK_sW_oP) = \underline{\sigma}(W_iK_sP_sW_i^{-1}) \geq \underline{\sigma}(K_s)\underline{\sigma}(P_s)\frac{1}{\kappa(W_i)},$$

where $\kappa(M) = \bar{\sigma}(M)/\underline{\sigma}(M)$ is the conditional number.

Small $\underline{\sigma}(K_s)$ might cause problem, yet, this can not happen
if $\underline{\sigma}(P_s)$ is large and α is not small.

Theorem

Any K_s guaranteeing stability radius $\alpha = 1/\gamma$ satisfies

$$\underline{\sigma}(K_s) \geq \frac{\underline{\sigma}(P_s) - \sqrt{\gamma^2 - 1}}{\sqrt{\gamma^2 - 1}\underline{\sigma}(P_s) + 1} \quad \text{if} \quad \underline{\sigma}(P_s) > \sqrt{\gamma^2 - 1}.$$

Corollary: If $\underline{\sigma}(P_s) >> \sqrt{\gamma^2 - 1}$ then $\underline{\sigma}(K_s) \geq \frac{1}{\sqrt{\gamma^2 - 1}}$.

Degradation at the high frequencies

At the high frequencies we need small $\underline{\sigma}(L_o)$ and $\underline{\sigma}(L_i)$. It is easy to verify that

$$\bar{\sigma}(L_o) = \bar{\sigma}(PW_iK_sW_o) = \bar{\sigma}(W_o^{-1}P_sK_sW_o) \leq \bar{\sigma}(P_s)\bar{\sigma}(K_s)\kappa(W_o),$$
$$\bar{\sigma}(L_i) = \bar{\sigma}(W_iK_sW_oP) = \bar{\sigma}(W_iK_sP_sW_i^{-1}) \leq \bar{\sigma}(K_s)\bar{\sigma}(P_s)\kappa(W_i),$$

Large $\bar{\sigma}(K_s)$ might cause problem, yet, this can not happen if $\bar{\sigma}(P_s)$ is small and α is not small.

Theorem

Any K_s guaranteeing stability radius $\alpha = 1/\gamma$ satisfies

$$\overline{\sigma}(K_s) \leq \frac{\sqrt{\gamma^2 - 1} + \overline{\sigma}(P_s)}{1 - \sqrt{\gamma^2 - 1}\overline{\sigma}(P_s)} \quad \text{if} \ \ \overline{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2 - 1}}.$$

Corollary: If $\overline{\sigma}(P_s) << 1/\sqrt{\gamma^2 - 1}$ then $\overline{\sigma}(K_s) \leq \sqrt{\gamma^2 - 1}$.

Interpretation for maximal stability radius α_{opt}

If α_{opt} is not small, i.e., is not far from 1, then

- the shapes of P_s and L_o are close at the low and at the high frequencies
- the proposed open loop shape can be achieved without loosing stability

The fact that $\alpha_{opt} << 1$ indicates that the shape of P_s is difficult to achieve and the constraints should be relaxed.

H_∞ loop shaping procedure

1. Choose W_i and W_o to shape $P_s = W_o P W_i$.
   There should be no unstable pole-zero cancellations in P_s.
   At this stage internal stability is not taken into account.
2. Compute normalized lcf for P_s and $\alpha_{opt} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \right\|_H^2}$. If $\alpha_{opt} << 1$ relax loop shaping requirements by adjusting the weights
3. If α_{opt} is acceptable select $\gamma > 1/\alpha_{opt}$ and synthesize stabilizing controller K_s that satisfies

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_\infty < \gamma.$$

4. Construct the final controller $K = W_i K_s W_o$.

Closed loop transfer matrices

Denote $\overline{\sigma}_i = \overline{\sigma}(W_i), \quad \underline{\sigma}_i = \underline{\sigma}(W_i), \quad \kappa_i = \kappa(W_i)$.

Theorem: Let P be the nominal plant and let $K = W_1 K_\infty W_2$ be the controller designed by loop shaping. Then if $b_{P_s,K_\infty} \geq 1/\gamma$ then

$$\begin{aligned}
\overline{\sigma}(K(I+PK)^{-1}) &\leq \gamma\overline{\sigma}(\tilde{M}_s)\overline{\sigma}_1\overline{\sigma}_2, \\
\overline{\sigma}((I+PK)^{-1}) &\leq \min\{\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, 1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}, \\
\overline{\sigma}(K(I+PK)^{-1}P) &\leq \min\{\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, 1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\
\overline{\sigma}((I+PK)^{-1}P) &\leq \frac{\gamma\overline{\sigma}(\tilde{N}_s)}{\underline{\sigma}_1\underline{\sigma}_2}, \\
\overline{\sigma}((I+KP)^{-1}) &\leq \min\{1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, \gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\
\overline{\sigma}(P(I+KP)^{-1}K) &\leq \min\{1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, \gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}
\end{aligned}$$

where

$$\begin{aligned}
\overline{\sigma}(\tilde{N}_s) &= \overline{\sigma}(N_s) = \left(\frac{\overline{\sigma}^2(P_s)}{1+\overline{\sigma}^2(P_s)}\right)^{1/2}, \\
\overline{\sigma}(\tilde{M}_s) &= \overline{\sigma}(M_s) = \left(\frac{1}{1+\overline{\sigma}^2(P_s)}\right)^{1/2}.
\end{aligned}$$

What did we study today?

- Normalized coprime factorization
- Robust stability subject to normalized lcf disturbance
  - Solution convenient for computation
  - Explicit formula for maximal stability radius
- H^∞ loop shaping procedure
  - Bounds on the degradation of open loop due to the introduction of stabilizing controller
  - Interpretation of the maximal stability radius