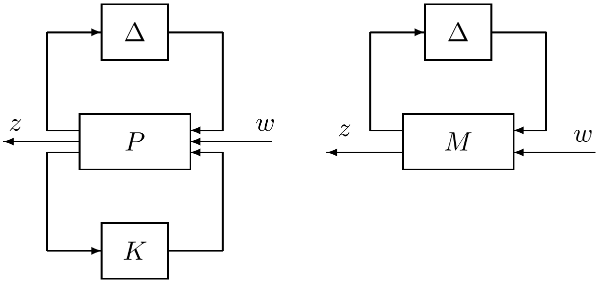
Structured singular value and μ-synthesis

Robust Control Course

Department of Automatic Control, LTH

Autumn 2011

LFT and General Framework

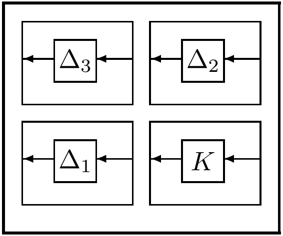


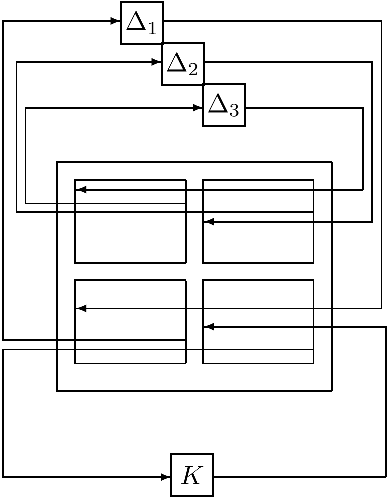
$$z = \mathcal{F}_u(\mathcal{F}_l(P, K), \Delta)w = \mathcal{F}_u(M, \Delta)w.$$

- Last week we considered the case with *full-block* uncertainty Δ
- Formulation with *block-diagonal* uncertainty may be useful
  - in problems with multiple uncertainty sources (obvious)
  - to include robust performance in formulation (clarified later)

Motivating example (multiple uncertainty sources)

Pulling out Uncertainties





Structured Uncertainty

- The pulled out uncertainty has a block-diagonal structure composed of primitive uncertain blocks.

- Every primitive block can be
  - unstructured matrix uncertainty $\Delta_i \in RH_\infty$
  - scalar uncertainties $\delta_i \in RH_\infty$ multiplied by identity matrices

- Thus, we shall assume that

$$\Delta(s) = \text{diag}\,\{\delta_1(s)I_{r_1}, \ldots, \delta_K(s)I_{r_K}, \Delta_1(s), \ldots, \Delta_L(s)\},$$

with $\|\delta_k\|_\infty \leq 1$ and $\|\Delta_l\|_\infty \leq 1$.

Remark: Uncertainty blocks $\delta_i(s)I_{r_i}$ sometimes stand for real parameter uncertainties covered (conservatively) with dynamic RH_∞ uncertainties

For the case with structured uncertainty:

- By Small Gain Theorem the condition $||M_{11}||_\infty < 1$ is sufficient for robust stability but *not necessary* for it.
  (Because the test ignores the known block structure.)

- Test for each uncertainty individually can be arbitrarily optimistic because it ignores interaction between the blocks.

Conclusion: We need to develop a new tool to deal with structured uncertainty.

Small gain theorem (reminder)

- The small gain theorem says that:

  $$(I - M\Delta)^{-1} \in RH_\infty,\ \forall\Delta \in \frac{1}{\gamma}\mathcal{B}RH_\infty$$

  $$\Leftrightarrow\ \|M\|_\infty = \sup_w \bar{\sigma}(M(j\omega)) < \gamma$$

  (Thus, we can refer $\|M\|_\infty^{-1} = 1/\gamma$ as the stability margin.)

- So if there exists $\Delta \in RH_\infty$ such that $(I - M\Delta)^{-1} \notin RH_\infty$, then $1/\gamma < \|\Delta\|$.

- Naturally we can consider the stability margin $\|M\|_\infty^{-1}$ as

  $$\|M\|_\infty^{-1} = \inf\{\|\Delta\|_\infty\ :\ (I - M\Delta)^{-1} \notin RH_\infty,\ \Delta \in RH_\infty\}$$

- Recalling the proof of the small gain theorem, we can formulate similar statements for each frequency ... (see the next slide)

Singular value - revisited

- Given $M \in \mathbb{C}^{p\times q}$, the following statement holds:

$$\det(I - M\Delta) \neq 0,\ \forall \Delta \in \alpha\mathcal{B}\mathbb{C}^{q\times p} \quad \Leftrightarrow \quad \bar{\sigma}(M) < 1/\alpha$$

(In these notations the "stability margin" is $\alpha \leftrightarrow 1/\gamma$)

- So if there exists Δ such that $\det(I - M\Delta) = 0$, then $\alpha < \|\Delta\|$.

- As before, the "stability margin" can be viewed as

$$\bar{\sigma}(M)^{-1} = \inf\{\|\Delta\| \ : \ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\}$$

- This gives an interesting perspective on the singular value.
  In fact, the singular value can be characterized as

$$\bar{\sigma}(M) = \frac{1}{\inf\{\|\Delta\| \ : \ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\}}$$

Structured singular value

Now consider the set of structured matrices

$$\mathbf{\Delta} \quad = \quad \{\text{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_K I_{r_K}, \Delta_1, \ldots, \Delta_L] \mid \delta_k \in C,\ \Delta_l \in C^{m_l \times m_l}\}$$

Definition: Given a matrix $M \in C^{p \times q}$ the *structured singular value* $\mu_{\mathbf{\Delta}}(M)$ is defined as

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\|\Delta\| \ : \ \det(I - M\Delta) = 0,\ \Delta \in \mathbf{\Delta}\}}.$$

If $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$ then $\mu_{\mathbf{\Delta}}(M) := 0$.

Elementary properties:

- $\mathbf{\Delta} = C^{q \times p} \Rightarrow \ \mu_{\mathbf{\Delta}}(M) = \bar{\sigma}(M)$.
- $\mathbf{\Delta} = \{\delta I \ : \ \delta \in C\} \Rightarrow \ \mu_{\mathbf{\Delta}}(M) = \rho(M)$.
- In general, $C \cdot I \subset \mathbf{\Delta} \subset C^{q \times p}$ so $\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \bar{\sigma}(M)$.

How good are the bounds?

Let

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}.$$

(1) For $M = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ with $\beta > 0$ we have

$$\rho(M) = 0, \quad \|M\| = \beta, \quad \mu_{\boldsymbol{\Delta}}(M) = 0.$$

(2) For $M = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$ we have

$$\rho(M) = 0, \quad \|M\| = 1.$$

Since $\det(I - M\Delta) = 1 + (\delta_1 - \delta_2)/2$ we get $\mu_{\boldsymbol{\Delta}}(M) = 1$.

Thus, both bounds are bad unless $\rho \approx \bar{\sigma}$.

Can we reduce the conservatism?

Invariant transformation

Let us try to find transformations which do not affect $\mu_{\mathbf{\Delta}}(M)$
but change $\rho(M)$ and $\bar{\sigma}(M)$.

Define two sets

$$\begin{aligned}\mathcal{U} &= \{U \in \mathbf{\Delta} \;:\; UU^* = I\},\\ \mathcal{D} &= \{\text{diag}[D_1, \ldots, D_K, d_1 I_{m_1}, \ldots, d_{L-1} I_{m_{L-1}}, I_{m_L}] :\\ &\qquad D_k \in C^{r_k \times r_k},\, D_k = D_k^* > 0,\, d_l \in R,\, d_l > 0\}.\end{aligned}$$

Note that for any $\Delta \in \mathbf{\Delta}$, $U \in \mathcal{U}$ and $D \in \mathcal{D}$ it holds

- $U^* \in \mathcal{U}$, $U\Delta \in \mathbf{\Delta}$, $\Delta U \in \mathbf{\Delta}$ (property of the set $\mathbf{\Delta}$).
- $\|U\Delta\| = \|\Delta U\| = \|\Delta\|$ (since $UU^* = I$).
- $D\Delta = \Delta D$ (property of the set $\mathcal{D}$).

Invariant transformation

Theorem

For all $U \in \mathcal{U}$ and $D \in \mathcal{D}$
1) $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(UM) = \mu_{\mathbf{\Delta}}(MU)$.
2) $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1})$.

Proof:

1) Since for each $U \in \mathcal{U}$

$$\begin{aligned} \det(I - M\Delta) = 0 \quad &\Leftrightarrow \quad \det(I - MUU^*\Delta) = 0 \\ \Delta \in \mathbf{\Delta} \quad &\Leftrightarrow \quad U^*\Delta \in \mathbf{\Delta} \end{aligned}$$

we get $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(MU)$.

2) For all $D \in \mathcal{D}$

$$\det(I - M\Delta) \quad = \quad \det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta)$$

since Δ and D commute. Therefore $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1})$.

Improving the bounds

At this point we can tighten the bounds as follows.

$$\sup_{U \in \mathcal{U}} \rho(UM) \leq \mu_{\boldsymbol{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \|DMD^{-1}\|$$

Theorem:

$$\sup_{U \in \mathcal{U}} \rho(UM) = \mu_{\boldsymbol{\Delta}}(M).$$

Theorem:

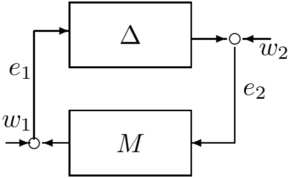
If $2K + L \leq 3$ then

$$\mu_{\boldsymbol{\Delta}}(M) = \inf_{D \in \mathcal{D}} \|DMD^{-1}\|.$$

Remarks:

- In general the quantity $\rho(UM)$ has many local maxima and the local search cannot guarantee to obtain $\mu(M)$.
- Computationally there is a slightly different formulation of the lower bound by Packard and Doyle which gives rise to a power algorithm. It usually works well but has no prove of convergence.
- The upper bound can be computed by convex optimization, but it is not always equal to $\mu(M)$ if $2K + L > 3$.
- It is the upper bound that is the cornerstone of μ synthesis, since it gives a sufficient condition for robust stability/performance.
- In Matlab use function **mu(M,blk)** to calculate the bounds of the structured singular value. See page 194 in the course book for more details.

Structured small gain theorem



Introduce the set

$$\mathcal{T}(\mathbf{\Delta}) = \{\Delta \in RH_\infty \ : \ \Delta(s) \in \mathbf{\Delta} \text{ for } s \text{ in RHP}\}.$$

The following result can be formulated.

Theorem

Let $M \in RH_\infty$. The closed-loop system (M, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{T}(\mathbf{\Delta})$ with $\|\Delta\|_\infty < 1$ if and only if

$$\sup_{\omega \in R} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq 1.$$

Structured small gain theorem - proof

The robust stability condition is

$$(I - M\Delta)^{-1} \in RH_\infty,\ \forall \Delta \in \mathcal{T}(\mathbf{\Delta}),\ \|\Delta\|_\infty < 1.$$

"$\Leftarrow$" By definition of structured singular value,

$$\det(I - M(s)\Delta(s)) \neq 0, \quad \forall s = iw.$$

However, we need to show this for all s in RHP. To this end, it is enough to notice that

- zeros of $\det(I - \alpha M\Delta)$ move continuously with respect to α
- for $\alpha < 1/\|M\|_\infty$, $\det(I - \alpha M\Delta)$ has no RHP zeros
- $\forall \alpha \leq 1$, $\det(I - \alpha M\Delta)$ has no imaginary zeros

"$\Rightarrow$" If $\sup_{\omega \in R} \mu_{\mathbf{\Delta}}(M(j\omega)) > 1$ then by definition of μ there exist ω_0 and Δ_0 with $\|\Delta_0\| < 1$ such that the matrix $\det(I - M(j\omega_0)\Delta_0) = 0$. Next, one can apply the same interpolation argument as in the Small Gain Theorem.

Structured small gain theorem - remarks

Remark: Unlikely the unstructured Small Gain Theorem the robust stability for all $\Delta \in \mathcal{T}(\mathbf{\Delta})$ with $\|\Delta\|_\infty \leq 1$ does not imply that

$$\sup_{\omega \in R} \mu_{\mathbf{\Delta}}(M(j\omega)) < 1.$$

It might be equal to 1. See example in [Zhou,p. 201].

Remark: The structured small gain theorem provides a tool for the analysis of robust stability subject to structured uncertainties.

Another useful result

Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be a complex matrix and suppose that $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are two defined structures which are compatible in size with M_{11} and M_{22} correspondingly.

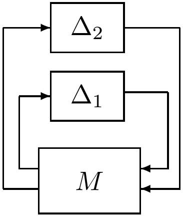
Introduce a third structure as $\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_1 & 0 \\ 0 & \boldsymbol{\Delta}_2 \end{bmatrix}$.

Theorem

$$1)\ \mu_{\boldsymbol{\Delta}}(M) < 1 \quad \Leftrightarrow \quad \left\{ \mu_{\boldsymbol{\Delta}_1}(M_{11}) < 1, \quad \sup_{\substack{\Delta_1 \in \boldsymbol{\Delta}_1 \\ \|\Delta_1\| \leq 1}} \mu_{\boldsymbol{\Delta}_2}(\mathcal{F}_u(M, \Delta_1)) < 1 \right\}$$

$$2)\ \mu_{\boldsymbol{\Delta}}(M) \leq 1 \quad \Leftrightarrow \quad \left\{ \mu_{\boldsymbol{\Delta}_1}(M_{11}) \leq 1, \quad \sup_{\substack{\Delta_1 \in \boldsymbol{\Delta}_1 \\ \|\Delta_1\| < 1}} \mu_{\boldsymbol{\Delta}_2}(\mathcal{F}_u(M, \Delta_1)) \leq 1 \right\}$$

Another useful result - proof



Proof: Prove only 1).

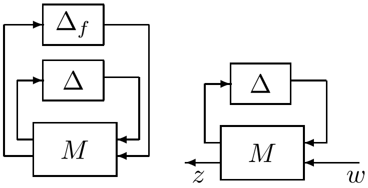
"$\Leftarrow$" Let $\|\Delta_i\| \leq 1$. By Schur complement

$$\det(I - M\Delta) = \det \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ -M_{21}\Delta_1 & I - M_{22}\Delta_2 \end{bmatrix} =$$
$$= \det(I - M_{11}\Delta_1)\det(I - \mathcal{F}_u(M, \Delta_1)\Delta_2) \neq 0.$$

"$\Rightarrow$" Basically the same identity plus (from definition of μ)

$$\mu_{\mathbf{\Delta}}(M) \geq \max\{\mu_{\mathbf{\Delta}_1}(M_{11}),\, \mu_{\mathbf{\Delta}_2}(M_{22})\}$$

Structured robust performance



Define an augmented block structure, where $p_2 \times q_2$ is the size of M_{22}.

$$\mathbf{\Delta}_P = \left[\begin{array}{cc} \mathbf{\Delta} & 0 \\ 0 & C^{q_2 \times p_2} \end{array} \right]$$

Theorem

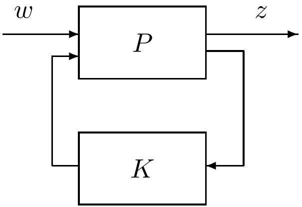
For all $\Delta \in \mathcal{T}(\mathbf{\Delta})$ with $\|\Delta\|_\infty < 1/\gamma$ the closed loop is well posed, internally stable and $\|\mathcal{F}_u(M, \Delta)\|_\infty \leq \gamma$ if and only if

$$\sup_{\omega \in R} \mu_{\mathbf{\Delta}_P}(M(j\omega)) \leq \gamma.$$

μ synthesis via $D - K$ iterations

The problem is to solve

$$\min_{K-\text{stab}} \|\mathcal{F}_l(P,K)\|_\mu.$$



Approximation for the upper bound

$$\min_{K-\text{stab}} \left(\inf_{D,\, D^{-1} \in H_\infty} \|D\mathcal{F}_l(P,K)D^{-1}\|_\infty \right)$$

under the condition $D(s)\Delta(s) = \Delta(s)D(s)$.

We try to solve this problem via $D - K$ iterations:

Step 1: Given D find K.
Step 2: Given K find D.

μ synthesis via $D - K$ iterations (contd.)

Remarks:

- Step 1 is the standard H_∞ optimization.
- Step 2 can be reduced to a convex optimization.
- No global convergence is guaranteed.
- Works sometimes in practice.

What have we learned today?

- Pulling out uncertainties leads to a diagonal structure
- Structured singular value μ is natural but is difficult to find
- Useful bounds of μ can be calculated
- Structured version of the small gain theorem
- Structured robust performance can be easily formulated
- Heuristic $D-K$ iterations as approach to μ synthesis.