Unstructured uncertainties and small gain theorem

Robust Control Course

Department of Automatic Control, LTH

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Introduction

Computable solutions of standard H_2 and H_∞ problems provide ready to use tools for the synthesis of MIMO controllers.

The resulting controllers, however, are not necessarily robust.

"Guaranteed Margins for LQG Regulators - they are none," J. C. Doyle, 1978

Recall that the purpose of robust control is that the closed loop performance should remain acceptable in spite of perturbations in the plant. Namely,

$$P_\Delta \approx P_0 \quad \Rightarrow \quad (P_\Delta, C) \approx (P_0, C),$$

where P_0 and P_Δ are the nominal and the perturbed plants.

Four kinds of specifications

Nominal stability
The closed loop is stable for the nominal plant P_0

(Youla/Kucera parameterization)

Nominal performance
The closed loop specifications hold for the nominal plant P_0

(Standard H_2 and H_∞ problems)

Robust stability
The closed loop is stable for all plants in the given set P_Δ

Robust performance
The closed loop specifications hold for all plants in P_Δ

Introduction

This lecture is dedicated to

- robust stability *(mainly)*
- robust performance *(a brief touch only)*

subject to unstructured uncertainties.

Our main tools will be

- small gain theorem *(later in this lecture)*
- H_∞ optimization *(previous lecture)*

One way to describe uncertainty

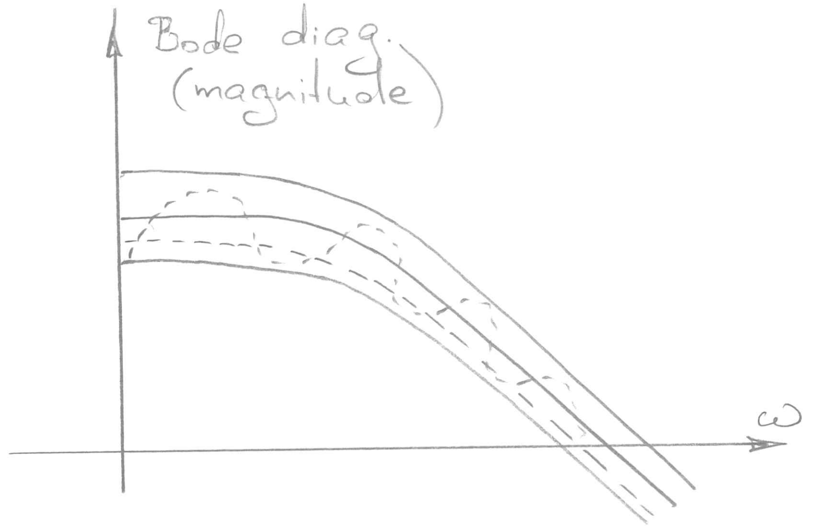
Additive uncertainty

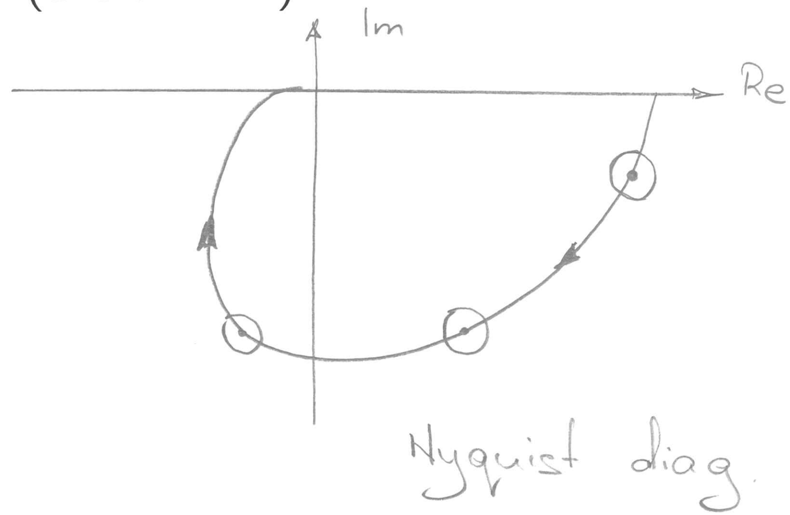
$$P_\Delta = P_0 + \Delta, \quad \Delta \in k \cdot \mathcal{B}RH_\infty,$$

where $\mathcal{B}RH_\infty$ is a ball in RH_∞, i.e.,

$$\mathcal{B}RH_\infty := \{G \in RH_\infty \ : \ ||G||_\infty \leq 1\}$$

Graphical interpretation of additive uncertainty (SISO case):





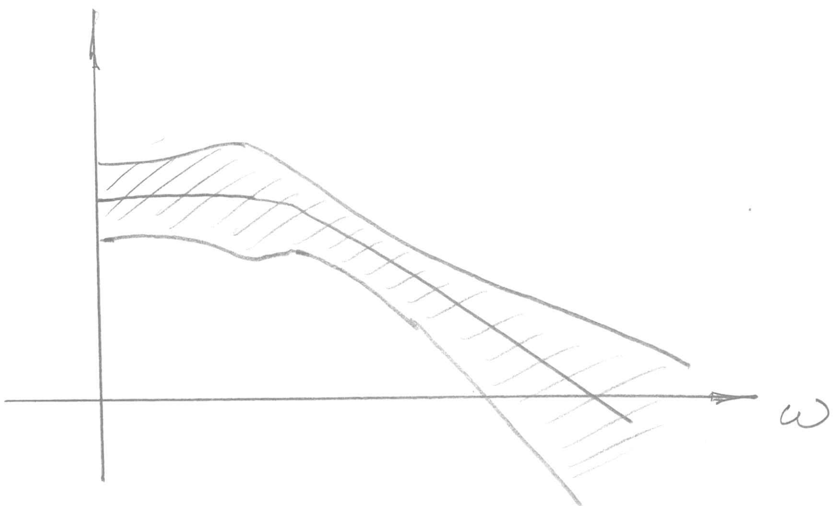
One way to describe uncertainty (contd.)

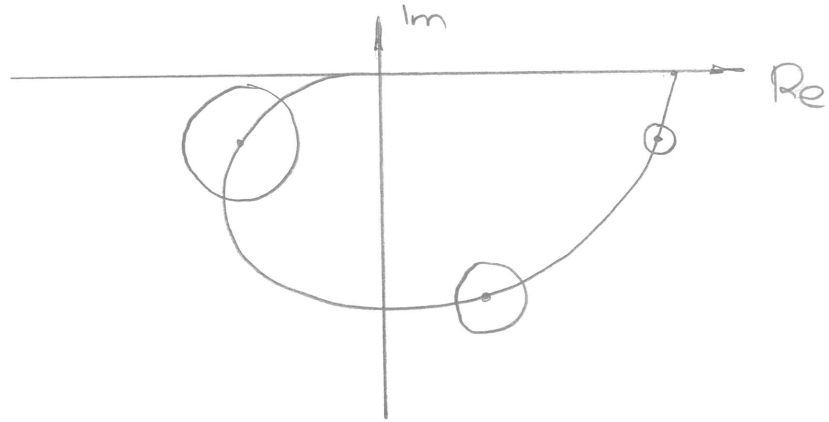
Additive uncertainty - more detailed weighted description

$$P_\Delta = P_0 + W_2 \Delta W_1, \quad \Delta \in \mathcal{B}RH_\infty.$$

- the weights define the uncertainty profile
- typically, $|W_{1/2}(iw)|$ are increasing functions of w
- choosing the weights may be a nontrivial task

Graphical interpretation of weighted additive uncertainty (SISO case):





Example 1

Consider a plant with parametric uncertainty

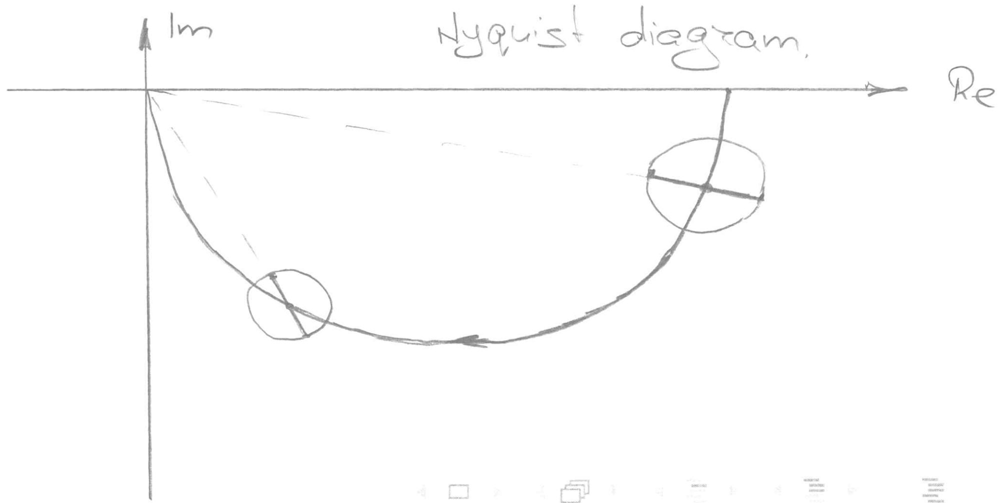
$$P(s) = \frac{1+\alpha}{s+1}, \quad \alpha \in [-0.2, 0.2].$$

It can be cast as a nominal plant with additive uncertainty

$$P_\Delta = \underbrace{\frac{1}{s+1}}_{P_0} + \underbrace{\frac{0.2}{s+1}}_{W}\Delta, \quad \Delta \in \mathcal{B}RH_\infty.$$

Note that this representation is conservative.

Graphical interpretation:



Example 2 (course book, page 133)

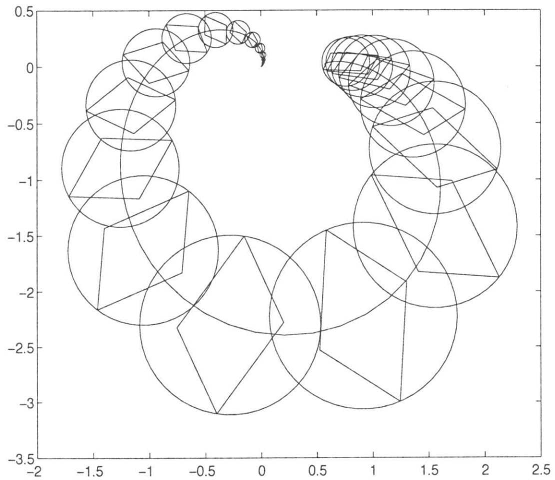
Consider a plant with parametric uncertainty

$$P(s) = \frac{10((2+0.2\alpha)s^2 + (2+0.3\alpha+0.4\beta)s + (1+0.2\beta))}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)},$$

for $\alpha, \beta \in [-1, 1]$. It can be cast as

$$P_\Delta = P_0 + W\Delta, \quad \Delta \in \mathcal{B}RH_\infty,$$

where $P_0 = P|_{\alpha,\beta=0}$ and $W = P|_{\alpha,\beta=1} - P|_{\alpha,\beta=0}$.



The small gain theorem

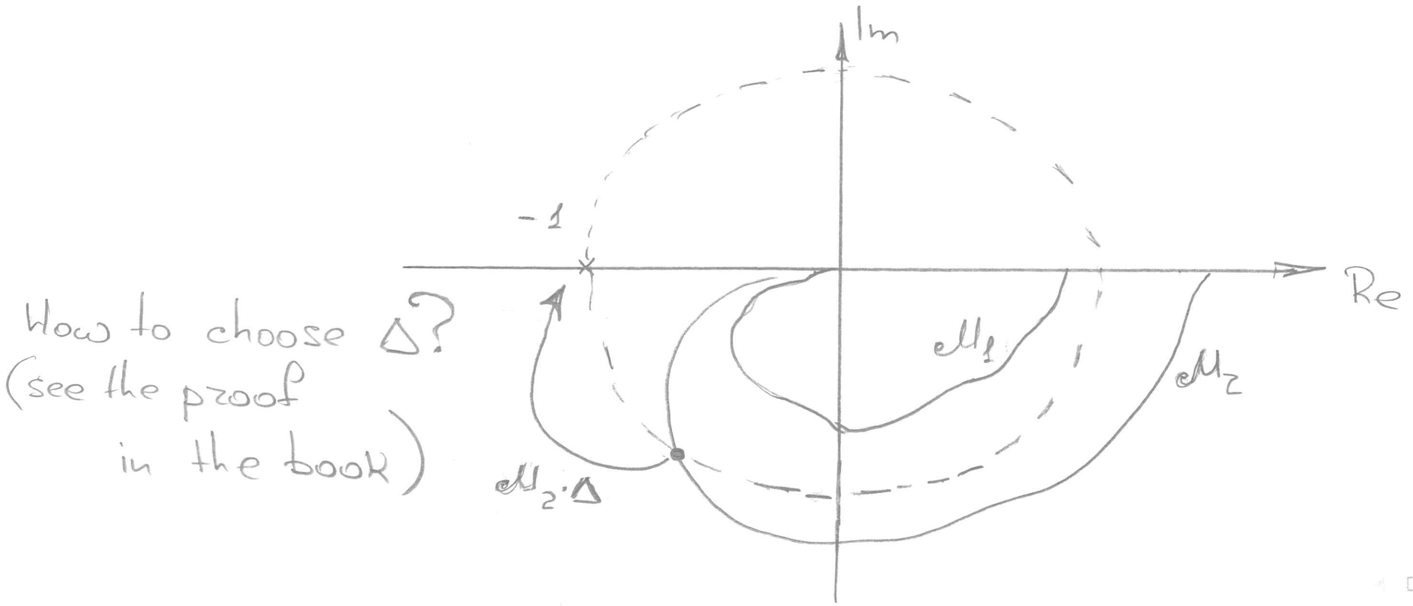
Theorem

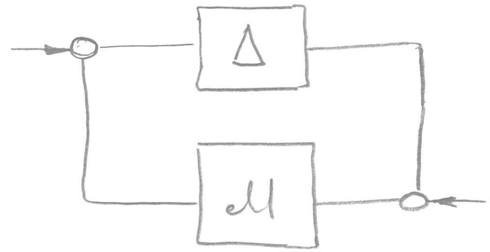
Suppose $M \in RH_\infty$. Then the closed loop system (M, Δ) is internally stable for all

$$\Delta \in \mathcal{B}RH_\infty := \{\Delta \in RH_\infty \mid \|\Delta\|_\infty \leq 1\}$$

if and only if $\|M\|_\infty < 1$.

Interpretation in terms of Nyquist criterion (SISO case):





Proof:

The internal stability of (M, Δ) is equivalent to

$$\begin{bmatrix} I & -\Delta \\ -M & I \end{bmatrix}^{-1} \in RH_\infty.$$

Since M, $\Delta \in RH_\infty$ it is equivalent to $(I - M\Delta)^{-1} \in RH_\infty$

([Zhou,Corollary 5.4]).

Thus we have to prove that $\|M\|_\infty < 1$ if and only if

$$(I - M\Delta)^{-1} \in RH_\infty, \quad \forall \Delta \in \mathcal{B}RH_\infty$$

Sufficiency:

Let $\|M\|_\infty < 1$ and $\Delta \in \mathcal{B}RH_\infty$.

Consider Neumann series decomposition $(I - M\Delta)^{-1} = \sum_{n=0}^{+\infty} (M\Delta)^n$.

Then $(I - M\Delta)^{-1} \in RH_\infty$, since $M\Delta \in RH_\infty$ and

$$\begin{aligned}
\|(I - M\Delta)^{-1}\|_\infty &\leq \sum_{n=0}^{+\infty} \|M\Delta\|_\infty^n \\
&\leq \sum_{n=0}^{+\infty} \|M\|_\infty^n = (1 - \|M\|_\infty)^{-1} < +\infty.
\end{aligned}$$

Necessity:

Fix $\omega \in [0, +\infty]$.

A constant $\Delta = \frac{\lambda M(j\omega)^*}{\|M(j\omega)\|}$ satisfies $\|\Delta\|_\infty \leq 1, \ \forall \lambda \in [0, 1]$.

As a result, we have that

$$\forall \lambda \in [0,1] \ : \ (I - M\Delta)^{-1} \in RH_\infty \ \Rightarrow \ \det\left(\frac{\|M\|}{\lambda} I - MM^*\right) \neq 0.$$

It gives $\|M\|^2 < \|M\|$ and, hence, $\|M\| < 1$.

The frequency is arbitrary, so we have $\|M\|_\infty < 1$.

$\square$

The small gain theorem - restatement

Obviously, the theorem can be reformulated as follows

Corollary

Suppose $M \in RH_\infty$. Then the closed loop system (M, Δ) is internally stable for all

$$\Delta \in \frac{1}{\gamma} \cdot \mathcal{B}RH_\infty := \{\Delta \in RH_\infty \mid \|\Delta\|_\infty \leq \frac{1}{\gamma}\}$$

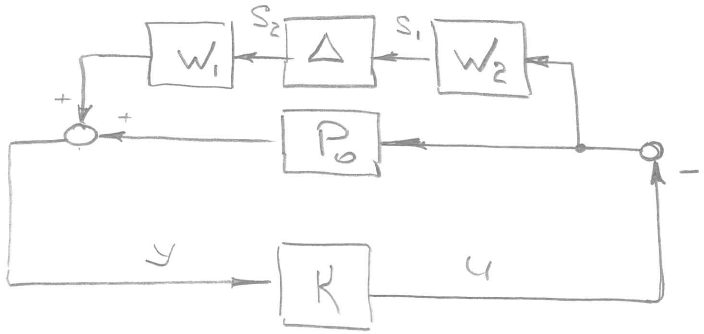
if and only if $\|M\|_\infty < \gamma$.

Once the H_∞ norm of M decreases,

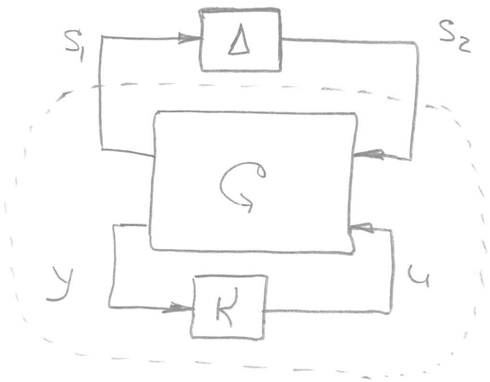
the radius of the admissible uncertainty increases.

Back to the control problem

Consider stabilization of a plant with additive uncertainty.



It can be represented in the following form.

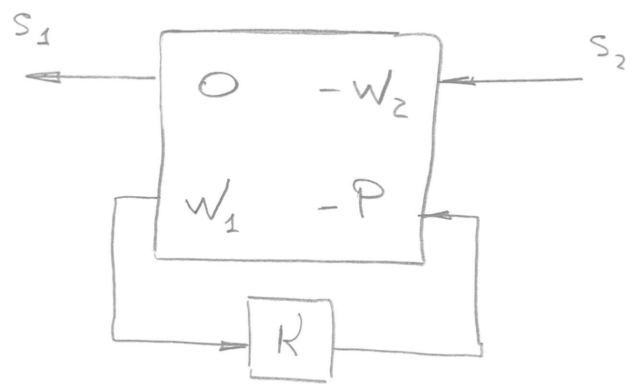


$$
G = \begin{array}{c} \\ (s_1) \\ (y) \end{array}
\begin{array}{c} \begin{array}{cc} (s_2) & (u) \end{array} \\ \begin{bmatrix} 0 & -W_2 \\ W_1 & -P \end{bmatrix} \end{array}
$$

This is a unified form for the stabilization problem with unstructured uncertainty.

Stabilization with additive uncertainties

Robust stabilization subject to additive uncertainty $P_\Delta = P_0 + W_1 \Delta W_2$ is equivalent to standard H_∞ optimization with:



This corresponds to the minimization of $||W_2 K S_o W_1||_\infty$.

- Minimizing the norm of the closed-loop system we maximize the radius of the admissible uncertainty
- Robust stabilization subject to additive uncertainty is an inherent part of the mixed sensitivity problem

Being slightly more formal, the following result can be formulated:

Theorem

Let $W_1, W_2 \in RH_\infty$, $P_\Delta = P_0 + W_1 \Delta W_2$ for $\Delta \in RH_\infty$ and K be a stabilizing controller for P_0. Then K is robustly stabilizing for all

$$\Delta \in \frac{1}{\gamma} \cdot \mathcal{B}RH_\infty$$

if and only if

$$\|W_2 K S_o W_1\|_\infty < \gamma.$$

Our next step will be to derive similar results

for different uncertainty descriptions ...

Basic Uncertainty Models

Additive uncertainty:

$$P_\Delta = P_0 + W_1 \Delta W_2, \;\; \Delta \in \mathcal{B}RH_\infty$$

Input multiplicative uncertainty:

$$P_\Delta = P_0(I + W_1 \Delta W_2), \;\; \Delta \in \mathcal{B}RH_\infty$$

Output multiplicative uncertainty:

$$P_\Delta = (I + W_1 \Delta W_2)P_0, \;\; \Delta \in \mathcal{B}RH_\infty$$

Inverse input multiplicative uncertainty:

$$P_\Delta = P_0(I + W_1 \Delta W_2)^{-1}, \;\; \Delta \in \mathcal{B}RH_\infty$$

Inverse output multiplicative uncertainty:

$$P_\Delta = (I + W_1 \Delta W_2)^{-1} P_0, \;\; \Delta \in \mathcal{B}RH_\infty$$

Feedback uncertainty:

$$P_\Delta = P_0(I + W_1 \Delta W_2 P_0)^{-1}, \; \Delta \in \mathcal{B}RH_\infty$$

Rcf uncertainty:

$$P_0 = NM^{-1}, \quad M, N \in RH_\infty \text{ and rcf}$$
$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}, \quad \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{B}RH_\infty$$

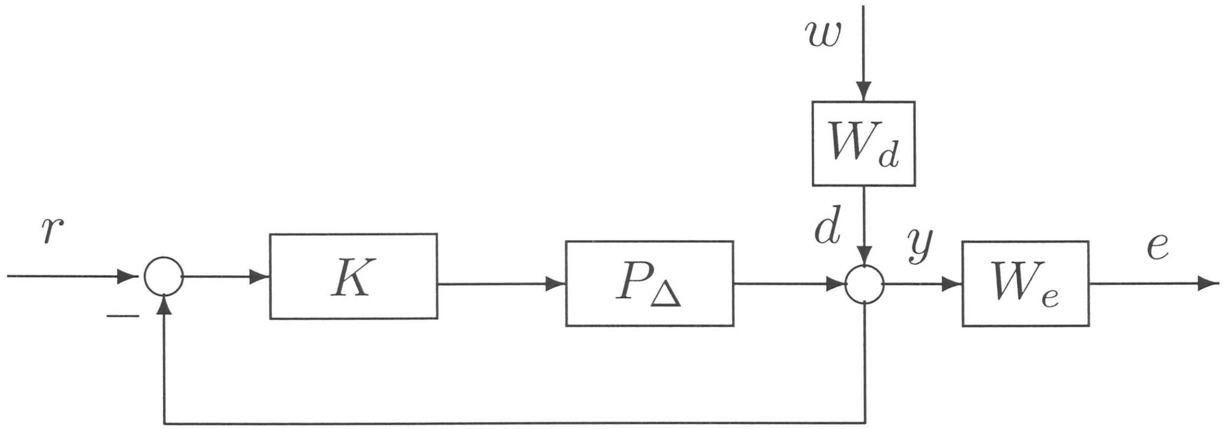
Lcf uncertainty:

$$P_0 = \tilde{M}^{-1}\tilde{N}, \quad \tilde{M}, \tilde{N} \in RH_\infty \text{ and lcf}$$
$$P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N), \quad \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \in \mathcal{B}RH_\infty$$

Robust stability tests for different uncertainty models

Uncertainty Model ($\Delta \in \frac{1}{\gamma}\mathcal{B}RH_\infty$)	Robust stability test
$(I + W_1 \Delta W_2) P_0$	$\|W_2 T_o W_1\|_\infty < \gamma$
$P_0 (I + W_1 \Delta W_2)$	$\|W_2 T_i W_1\|_\infty < \gamma$
$(I + W_1 \Delta W_2)^{-1} P_0$	$\|W_2 S_o W_1\|_\infty < \gamma$
$P_0 (I + W_1 \Delta W_2)^{-1}$	$\|W_2 S_i W_1\|_\infty < \gamma$
$P_0 + W_1 \Delta W_2$	$\|W_2 K S_o W_1\|_\infty < \gamma$
$P_0 (I + W_1 \Delta W_2 P_0)^{-1}$	$\|W_2 S_o P W_1\|_\infty < \gamma$
$(\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ $\Delta = [\tilde{\Delta}_N \ \tilde{\Delta}_M]$	$\left\| \begin{bmatrix} K \\ I \end{bmatrix} S_o \tilde{M}^{-1} \right\|_\infty < \gamma$
$(N + \Delta_N)(M + \Delta_M)^{-1}$ $\Delta = [\Delta_N \ \Delta_M]'$	$\left\| M^{-1} S_i \begin{bmatrix} K & I \end{bmatrix} \right\|_\infty < \gamma$

Robust performance with output multiplicative uncertainty



Let T_{ew} be the closed loop transfer function from w to e. Then

$$T_{ew} = W_e(I + P_\Delta K)^{-1} W_d.$$

Given robust stability, a robust performance specification is

$$\|T_{ew}\|_\infty < 1, \quad \forall \Delta \in \mathcal{B}RH_\infty$$

This can be written as

$$\|W_e S_o (I + W_1 \Delta W_2 T_o)^{-1} W_d\|_\infty < 1, \qquad \forall \Delta \in \mathcal{B}RH_\infty$$

SISO case

Consider for simplicity a case when K and P_0 are scalar. Then we can join W_e and W_d as well as W_1 and W_2 to get RP condition

$$\|W_T T\|_\infty < 1, \quad \left\| \frac{W_S S}{1 + \Delta W_T T} \right\|_\infty < 1$$

for all $\Delta \in \mathcal{B}RH_\infty$.

Theorem: A necessary and sufficient condition for RP is

$$\| |W_S S| + |W_T T| \|_\infty < 1.$$

Proof: The condition $\| |W_S S| + |W_T T| \|_\infty < 1$ is equivalent to

$$\|W_T T\|_\infty < 1, \quad \left\| \frac{W_S S}{1 - |W_T T|} \right\|_\infty < 1.$$

Proof

"$\Leftarrow$"

At any point $j\omega$ it holds

$$1 = |1 + \Delta W_T T - \Delta W_T T| \le |1 + \Delta W_T T| + |W_T T|$$

hence $1 - |W_T T| \le |1 + \Delta W_T T|$. This implies that

$$\left\| \frac{W_S S}{1 + \Delta W_T T} \right\|_\infty \le \left\| \frac{W_S S}{1 - |W_T T|} \right\|_\infty < 1.$$

"$\Rightarrow$"

Assume robust performance. Pick a frequency ω where $\frac{|W_S S|}{1 - |W_T T|}$ is maximal. Now pick Δ so that $1 - |W_T T| = |1 + \Delta W_T T|$ at this point ω. We have

$$\left\| \frac{W_S S}{1 - |W_T T|} \right\|_\infty = \frac{|W_S S|}{1 - |W_T T|} = \frac{|W_S S|}{|1 + \Delta W_T T|} \le \left\| \frac{W_S S}{1 + \Delta W_T T} \right\|_\infty \le 1.$$

Robust Performance for Unstructured Uncertainty

Remarks:

- Note that the condition for nominal performance is $\|W_S S\|_\infty < 1$, while the condition for robust stability is $\|W_T T\|_\infty < 1$. Together the two conditions say something about robust performance:

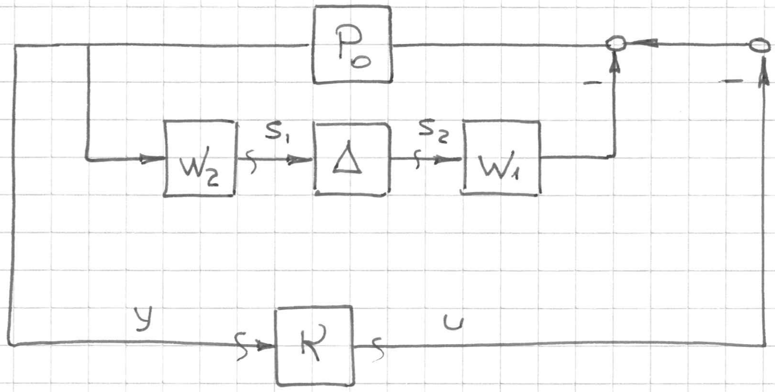
$$\begin{aligned} \max\{|W_S S|, |W_T T|\} &\leq |W_S S| + |W_T T| \leq \\ &\leq 2\max\{|W_S S|, |W_T T|\} \end{aligned}$$

- For MIMO systems the corresponding condition for robust performance becomes only sufficient (see [Zhou,p. 149]).
- It is possible to obtain robust performance conditions for other uncertainty models as well. Some of them are simple others are very messy.

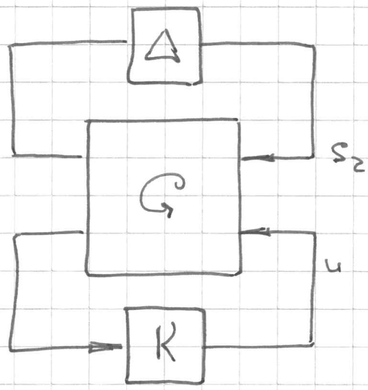
What did we study today?

- Standard ways to describe uncertainty
- Small gain theorem
- The use of small gain theorem:
  the idea to form LFT by pulling out the uncertainty
- Robust performance criteria (for a special case)

Derivations for the case of feedback uncertainties.



$$\Updownarrow$$



$$G = \begin{array}{cc} (s_2) & (u) \\ \end{array}$$
$$G = \begin{bmatrix} -W_2 P_o W_1 & -W_2 P_o \\ -P_o W_1 & -P_o \end{bmatrix}$$

$$\mathcal{F}_l\left(\begin{bmatrix} -W_2 P_o W_1 & -W_2 P_o \\ -P_o W_1 & -P_o \end{bmatrix}, K\right) = -W_2 S_o P_o W_1$$

Thus, the problem is equivalent to minimization of

$$\| W_2 S_o P_o W_1 \|_\infty$$