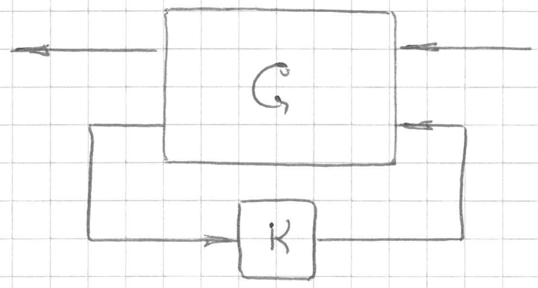
Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems

Consider the generalized plant setup,



where G is given by its minimal state-space realization:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Our aim is to find a proper K that guarantees internal stability and minimizes $\|\mathcal{F}_l(G,K)\|$, where $\|\cdot\|$ is either $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm.

Standard assumptions and their interpretation.

$\underline{\mathcal{A}_1}$: $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ has no unstable hidden modes

$\underline{\mathcal{A}_2}$: $\left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right]$ has no invariant zeros in $j\mathbb{R}$

and $D_{12}' D_{12} = I$.

$\underline{\mathcal{A}_3}$: $\left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]$ has no invariant zeros in $j\mathbb{R}$

and $D_{21} D_{21}' = I$

Interpretation of $\mathcal{A}_1$:

It is equivalent to saying that

- $(\mathcal{A}, B_2)$ is stabilizable
- $(\mathcal{A}, C_2)$ is detectable

This is a necessary and sufficient condition for stabilizability.

Interpretation of $\mathcal{A}_2$:

It is needed for well-posedeness of the optimization problem. (Necessary for well-posedeness of the Riccati-based solution.)

<u>Example:</u> Consider the generalized plant

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s}{s+1} \\ 1 & 0 \end{bmatrix}$$, which doesn't satisfy $\mathcal{A}_2$.

It corresponds to minimization of

$$\left\| \frac{1}{s+1} + \frac{s}{s+1} K \right\|.$$

Clearly $K_* = -\frac{1}{s}$ would nullify this expression, but it is not stabilizing.

A stabilizing $K = \frac{1}{s+\mathcal{E}}$, $\mathcal{E} > 0$ can approach K_* arbitrary close, but never reaches it.

So there is no optimal solution.

The fact that $D_{12}' D_{12} = I$ is just the matter of normalization.

To normalize we just need to redefine the design parameter as

$$K_{new} = \left(D_{12}' D_{12}\right)^{1/2} K$$

Interpretation of $\mathcal{A}_3$: similar as $\mathcal{A}_2$

Reduction to model matching

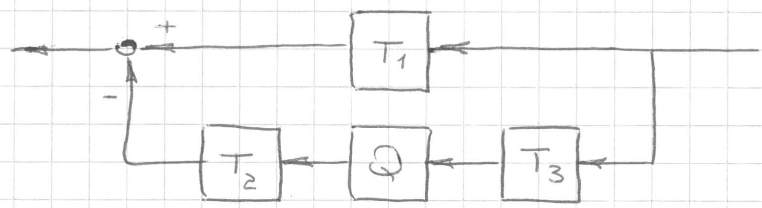
Applying Youla/Kučera parameterization of stabilizing solutions, the problem

$$\min_{\text{stab. } K} \| \mathcal{F}_l(G, K) \|$$

is reduced to

$$\min_{Q \in H_\infty} \| T_1 - T_2 Q T_3 \| .$$

The latter problem is usually called <u>model matching</u>.



The model T_1 has to be matched by the lower branch.

There always exist transfer matrices U_2 and U_3 satisfying $U_2^{\sim} U_2 = I$ and $U_3 U_3^{\sim} = I$, such that

$$U_2 T_2 = \begin{bmatrix} \dot{T}_2 \\ 0 \end{bmatrix} \quad , \ \dot{T}_2 \text{ is square and invertible}$$

$$T_3 U_3 = \begin{bmatrix} \dot{T}_3 & 0 \end{bmatrix} \quad , \ \dot{T}_3 \text{ is square and invertible}$$

$$U_2 T_1 U_3 = \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{21} & \dot{T}_{22} \end{bmatrix}$$

Pre- and post-multiplication by U_2 and U_3 doesn't change neither H_2 nor H_∞ norm. (Easy to check using norm definition.)

As a result, multiplying

$$U_2 \times \qquad \| T_1 - T_2 Q T_3 \| \qquad \times U_3 \quad ,$$

we can reformulate our problem as

$$\min_{Q \in H_\infty} \left\| \begin{bmatrix} \dot{T}_{11} - \dot{T}_2 Q \dot{T}_3 & \dot{T}_{12} \\ \dot{T}_{21} & \dot{T}_{22} \end{bmatrix} \right\|$$

- If all the blocks in the above formulation are not void, then the problem is referred to to as a <u>four-block problem</u>.

- If either $\dot{T}_{12}, \dot{T}_{22}$ or $\dot{T}_{21}, \dot{T}_{22}$ are void, then the problem is referred to as <u>two-block prob.</u>

> This happens if either G_{12} or G_{21} and, as a result, either T_2 or T_3 are square. Practically, this is a situation for
>
> - State feedback problem
> - Standard estimation problem
> - Standard feedforward tracking
> - the list can be continued ...

- If all $\dot{T}_{12}, \dot{T}_{21}$ and $\dot{T}_{22}$ are void, then the problem is referred to as <u>one-block problem</u>.

> This happens if both G_{12} and G_{21} are square.

Remark:

The H_2 norm has a „nice property":

$$\left\| \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \right\|_2^2 = \|G_{11}\|_2^2 + \|G_{12}\|_2^2 + \|G_{21}\|_2^2 + \|G_{22}\|_2^2$$

As a result, the H_2 problem automatically reduces to one-block optimization:

$$\min_{Q \in H_\infty} \| \grave{T}_{11} - \grave{T}_2 Q \grave{T}_2 \|_2$$

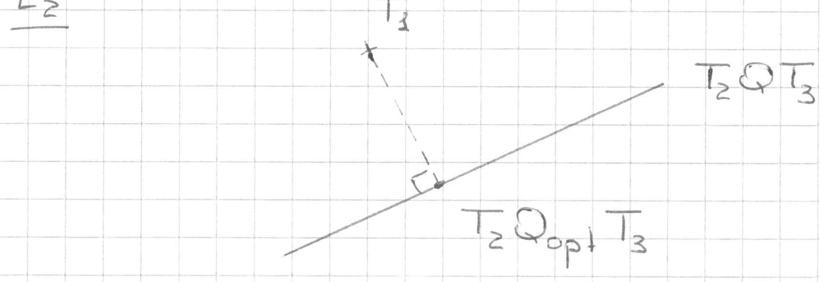
(Note, however, that this is not true for the H_∞ case.)

Derivation of the H_2 optimal solution

$$Q_{opt} = \underset{Q \in H_2}{\operatorname{argmin}} \| T_1 - T_2 Q T_3 \|_2$$

$$\boxed{\text{It can be assumed without loss of the generality that } T_1 \in L_2, \; Q \in H_2}$$

The Hilbert space arguments can be applied now



We have that Q_{opt} is the optimal solution iff

$$\langle T_1 - T_2 Q_{opt} T_3 , T_2 Q T_3 \rangle = 0 \quad , \forall Q \in H_2$$

$$\boxed{\begin{array}{l}\text{In the } L_2 \text{ space adjoint = conjugate,} \\ \text{namely,} \\ \langle G_1, G_2 G_3 G_4 \rangle = \langle G_2^\sim G_1 G_4^\sim, G_3 \rangle\end{array}}$$

$$\langle T_2^\sim T_1 T_3^\sim - T_2^\sim T_2 Q_{opt} T_3 T_3^\sim , Q \rangle = 0 , \quad \forall Q \in H_2$$

$$T_2^\sim T_1 T_3^\sim - T_2^\sim T_2 Q_{opt} T_3 T_3^\sim \in H_2^\perp$$

Introduce spectral and co-spectral factorizations:

$$S_2^\sim S_2 = T_2^\sim T_2 \quad , \quad S_3 S_3^\sim = T_3 T_3^\sim .$$

Substituting, we have

$$T_2^\sim T_1 T_3^\sim - S_2^\sim \underbrace{S_2 Q_{opt} S_3}_{\overline{Q}_{opt}} S_3^\sim \in H_2^\perp$$

- since S_2, S_3 are bistable we can absorb them into design parameter.
- since $S_2^\sim, S_3^\sim$ are „anti-bistable", we can pre- and post-multiply by $(S_2^\sim)^{-1}, (S_3^\sim)^{-1}$.

$$S_2^{-\sim} T_2^\sim T_1 T_3^\sim S_3^{-\sim} - \overline{Q}_{opt} \in H_2^\perp$$

$$\overline{Q}_{opt} = \left(S_2^{-\sim} T_2^\sim T_1 T_3^\sim S_3^{-\sim} \right)_+$$

$$\boxed{Q_{opt} = S_2^{-1} \left(S_2^{-\sim} T_2^\sim T_1 T_3^\sim S_3^{-\sim} \right)_+ S_3^{-1}}$$

Using the state-space machinery, explicit state-space formulae for the solution can be derived.
Two spectral factorizations result in two AREs.

§13.5 State-space solution for the H_2 problem

Let A_{1-3} hold, then the unique H_2 optimal controller

$$K_{opt} = -\left[\begin{array}{c|c} A + B_2 F + L C_2 & L \\ \hline F & 0 \end{array}\right]$$

attains optimal performance

$$\gamma_{opt} = \| \mathcal{F}_1(G, K_{opt}) \|_2 = \sqrt{tr(B_1' X B_1) + tr(F Y F')}$$

$$= \sqrt{tr(C_1' Y C_1) + tr(L' X L)} ,$$

where $F = -B_2' X - D_{12}' C_1$, $L = -Y C_2' - B_1 D_{21}'$ and

$$X A + A' X + C_1' C_1 - (B_2' X + D_{12}' C_1)' (B_2' X + D_{12}' C_1) = 0$$

$$A Y + Y A' + B_1 B_1' - (Y C_2' + B_1 D_{21}') (Y C_2' + B_1 D_{21}')' = 0$$

- In Matlab you may use function „h2syn".

The H_∞ problem

- Reduction to one-block problem is not trivial. (unlike the H_2 case)

- Hilbert space arguments are not applicable.

- The problem can be reduced to a Nehari problem:

$$\min_{F \in RH_\infty} \| R - F \|_\infty \quad \text{for some } R \in L_\infty .$$

Solved by Nehari in 1957.

- Let's proceed directly to the state-space result, where only suboptimal problem is tractable.

§14.2 State-space solution for the H_∞ problem.

Let $\mathcal{A}_{1-3}$ hold, then there exist internally stabilizing K such that $\| \mathcal{F}_l(G, K) \|_\infty < \gamma$ for some given $\gamma > 0$ iff:

1. There exists stabilizing $X \geq 0$ solving ARE

$$X\mathcal{A} + \mathcal{A}'X + C_1'C_1 + \gamma^{-2} X B_1 B_1' X - (B_2'X + D_{12}'C_1)'(B_2'X + D_{12}'C_1) = 0$$

2. There exists stabilizing $Y \geq 0$ solving ARE

$$\mathcal{A}Y + Y\mathcal{A}' + B_1B_1' + \gamma^{-2} Y C_1'C_1 Y - (YC_2' + B_1D_{21}')(YC_2' + B_1D_{21}')' = 0$$

3. $\rho(XY) < \gamma^2$

if these conditions hold, then one γ-suboptimal stabilizing solution is

$$K_\gamma(s) = \left[\begin{array}{c|c} A + \gamma^{-2} B_1 B_1' X + B_2 F + ZL\left(C_2 + \gamma^{-2} D_{21} B_1' X\right) & ZL \\ \hline F & 0 \end{array}\right]$$

where $Z = \left(I - \gamma^{-2} YX\right)^{-1}$ well defined.

- The optimal solution can be found using bisection algorithm.
- In Matlab you may use „hinfsyn" command.

So where are we and where are we going?

We have explicit and numerically implementable solutions for the standard H_2 and H_∞ problems.

- provides convenient tools for MIMO synthesis.

Things to remember:

- no criterion can reflect all our requirements.
- „optimal" might have nothing to do with „good."
  (optimal is rather close to extreme)
- more comprehensive cost function results in less transparent solution.
- optimization should be used as a tool rather then as a goal.

Natural questions to ask:

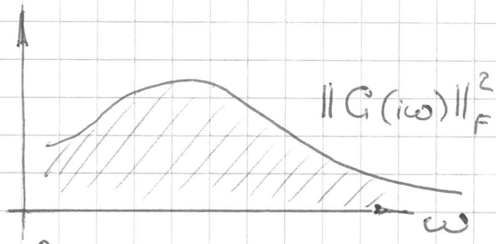
- How to choose the norm and how to interpret its minimization?
- How to choose the weights?
- How to relate and to integrate the optimization tools with our previous knowledge from classical methods?

Let's try to clarify these questions to some extent...

* Interpretation of the H_2 norm minimization

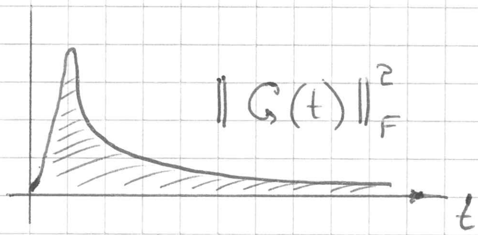
- Reminder: Interpretation of the H_2 norm.

Frequency domain -



„energy" of the frequency response.

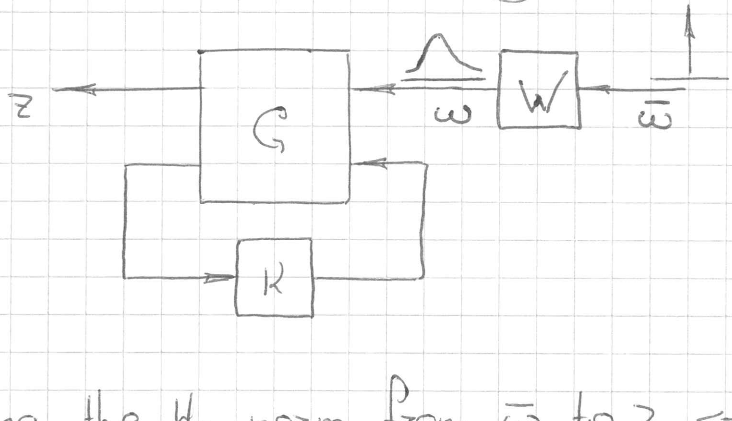
Time domain -



„energy" of the impulse response.

- Minimizing $\| \mathcal{F}_1(G, K)\|_2$, we minimize the energy of the impulse response of the regulated signal.
- And what if the shape of the external signal is known to some extent?

We can approximate this shape by a response of a linear system and then introduce this linear system as a weight.



Minimizing the H_2 norm from $\bar{\omega}$ to z <=> Minimizing the energy of the response of z to ⌒ in ω.

- This provides additional interpretation for the weights in H_2 control.

- And suggests that H_2 optimization is especially appropriate if something is known about the shape of external signals.

* LQR problem as a special case of H_2 opt.

- Given $\begin{cases} \dot{x} = Ax + Bu \\ y = x \end{cases}$ with i.c. $x(0) = x_0$

minimize $J = \int_0^{\infty} (x'Qx + u'Ru)\, dt$

for $Q \geq 0$ and $R > 0$.

- Two things to notice:

1). $J = \|z\|_2^2$ for $z = \begin{bmatrix} Q^{1/2} x \\ R^{1/2} u \end{bmatrix}$

2). The system can be rewritten as

$$\begin{cases} \dot{x} = Ax + x_0 \delta(t) + Bu(t) \quad , \quad x(0) = 0 \\ y = x \end{cases}$$

Dirac delta

- So this is a special case of a standard H_2 problem with

$G = $

* Kalman filter as a special case of H_2 opt.

- Consider
$$\begin{cases} \dot{x}(t) = A x(t) + v_x(t) \\ y(t) = C x(t) + v_y(t) \end{cases}$$

where $v_x(t)$, $v_y(t)$ are white Gaussian zero-mean noises with

$$E\left(v_x(t)\, v_x'(\tau)\right) = Q_x\, \delta(t-\tau), \quad Q_x \geq 0$$
$$E\left(v_y(t)\, v_y'(\tau)\right) = Q_y\, \delta(t-\tau), \quad Q_y > 0$$

Basing on the measurement of y we need to estimate x with $\hat{x}$. The cost function is

$$J = tr\left(E\left[(x-\hat{x})(x-\hat{x})'\right]\right)$$

- Two things to notice:

1). There is anology between stochastic and deterministic approaches.

	input	output	gain
Determ.	$\delta(t)$	$\lVert y \rVert_2$	H_2 norm
Stoch.	w.g.n. (*)	$tr(E(yy'))$	H_2 norm

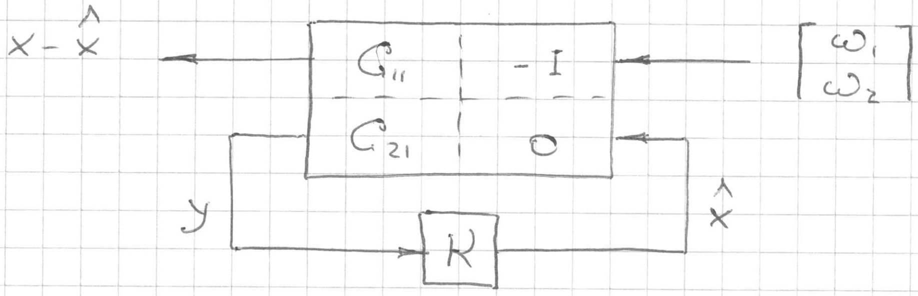
(*) white Gaussian zero-mean noise with unit variance.

2). We can represent $v_x = Q_x^{1/2} \omega_1$, $v_y = Q_y^{1/2} \omega_2$ for ω_1, ω_2 like in (*)

So the problem can be described now by

$$\begin{cases} \dot{x} = Ax + \begin{bmatrix} Q_x^{1/2} & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \\ y = Cx + \begin{bmatrix} 0 & Q_y^{1/2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \end{cases}$$

and can be formulated as a special case of the standard H_2 problem



with $G_{11} = \left[\begin{array}{c|cc} A & Q_x^{1/2} & 0 \\ \hline 1 & 0 & 0 \end{array}\right]$, $G_{21} = \left[\begin{array}{c|cc} A & Q_x^{1/2} & 0 \\ \hline C & 0 & Q_y^{1/2} \end{array}\right]$

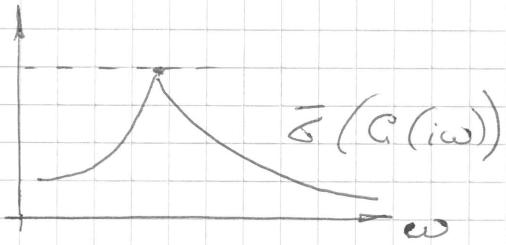
so that

$$G = \begin{bmatrix} G_{11} & -1 \\ G_{21} & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & Q_x^{1/2} & 0 \\ \hline 1 & 0 & 0 \\ C & 0 & Q_y^{1/2} \end{array}\right]$$

* Interpretation of the H_∞ norm minimization

- Reminder: Interpretation of the H_∞ norm

Frequency domain



„peak" of the frequency response

Time domain
$$\|G(s)\|_\infty = \sup_{\|u\|_2 = 1} \|G(s)u\|_2$$

gain with respect to $\|\cdot\|_2$ signal norm

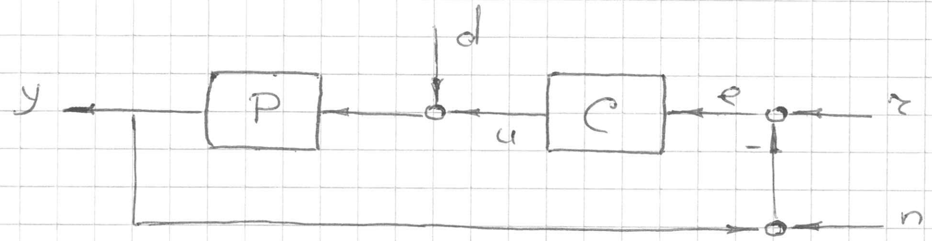
- Minimizing $\|F_l(G,K)\|_\infty$, we minimize the energy of the regulated signal for the worst case of the external signal. (There exists a game-theoretic approach to H_∞.)

- One way to choose weights is by reflecting
  - relative importance of regulated outputs at different frequencies.
  - expected amplitude of external signals at different frequencies.

The, so called, „mixed sensitivity" problem is relevant in this context.

§6

* Consider a single-loop case



$$L_i = CP \qquad\qquad L_o = PC$$

$$S_i = (1+L_i)^{-1} \qquad\qquad S_o = (1+L_o)^{-1}$$

$$T_i = 1 - S_i \qquad\qquad T_o = 1 - S_o$$

Roughly speaking, our requirements during the controller synthesis are:

1) internal stability

2) $|S(j\omega)|$ small at the low frequency

- for tracking and disturbance attenuation

3) $|T(j\omega)|$ or $|C(j\omega)S(j\omega)|$ not too large at the low frequencies.

- to limit the control effort

4) $|T(j\omega)|$ or $|C(j\omega)S(j\omega)|$ small at the high frequencies.

- for noise rejection

5) $|S(j\omega)|, |T(j\omega)|$ should not have large peaks.

While trying to achieve these properties, we should remember that:

1). $|S(j\omega)|$ and $|T(j\omega)|$ can not be small at the same frequency, since $S+T=I$

2). $|S(j\omega)|$ can not be less then 1 at all frequencies.

Bode sensitivity integral

$$\int_0^{\infty} \ln|S(j\omega)|\,d\omega = \pi \cdot \sum_{i=1}^{m} Re(p_i) \geq 0$$

p_i, $i=1..m$ are unstable poles of P

We are going to translate the reasoning above to the „optimization language", casting controller synthesis to a standard problem.

§6.2 * Weighted sensitivity problem

Consider the requirement

$$S_o(i\omega) < \begin{cases} \varepsilon_\sigma, & \forall \omega \leq \omega_o \\ 1+\delta_\sigma, & \forall \omega > \omega_o \end{cases} \qquad (1)$$

- ω_o determines frequency content of r and d

- $0 < \varepsilon_\sigma < 1$ quantifies the requirement on $|S|$ to be small.

- $0 < \delta_\sigma$ to omit large peaks of $|S|$

To formalize the requirement in (1) introduce stable transfer function $W_\sigma(s)$, such that

$$W_\sigma(i\omega) = \begin{cases} 1/\varepsilon_\sigma, & \forall \omega \leq \omega_0 \\ \dfrac{1}{1+\delta_\sigma}, & \forall \omega > \omega_0 \end{cases}$$

Now we can rewrite (1) as $|W_\sigma(j\omega)S(j\omega)| < 1$, $\forall\omega$, or equivalently as

$$\| W_\sigma(s) S_0(s) \|_\infty < \gamma = 1$$

The problem of synthesis that admits the requirement (1) can now be handled via solving

$$\gamma_{opt} = \inf_{\text{stab. } C} \| W_\sigma(s) S(s) \|_\infty \qquad (2)$$

and checking if $\gamma_{opt} \leq 1$.

The optimization in (2) is refered to as <u>weighted sensitivity problem</u>. It is a special case of a standard problem with:

$$G(s) = \left[\begin{array}{c|c} W_\sigma & -W_\sigma P \\ \hline I & -P \end{array}\right]$$

Although $W_\delta(s)$ introduced above is infinite-dimensional, it can be approximated by a finite-dimensional system. First- and second-order approximations could be:

$$W_{\delta,1} = \frac{\frac{1}{1+\delta_\delta} s + \frac{1}{\varepsilon_\delta}\omega_a}{s+\omega_a}$$

$$W_{\delta,2} = \frac{\frac{1}{1+\delta_a} s^2 + \sqrt{\frac{2}{\varepsilon_\delta(1+\delta_\delta)}}\,\omega_a s + \frac{1}{\varepsilon_\delta}\omega_o^2}{s^2 + \sqrt{2}\,\omega_o s + \omega_o^2}$$

Some remarks are in order:

1). γ_{opt} is treated as an „indicator" rather then as the achievable performance.

If γ_{opt} is small, then we can tighten the requirements.

If $\gamma_{opt} > 1$, then the requirement can not be achieved and should be relaxed.

So the optimization serves as a tool.

2). After the performance requirements have been quantified, the solution is a technical step! Internal stability is granted.
Negative answer ($\gamma_{opt} > 1$) means that the requirement is not achievable. (Unlike in the classical methods.)

§6.2

* Mixed sensitivity problems

Weighted sensitivity problem captures only a part of the picture. In particular, it doesn't account for the control signal.

It is natural to complement our problem formulation with the requirement

$$|T_o(j\omega)| < \begin{cases} 1+\delta_\tau, & \forall \omega < \omega_1 \\ \varepsilon_\tau \left(\omega_1/\omega\right)^\nu, & \forall \omega > \omega_1 \end{cases}$$

- ω_1 - frequency content of the noise
- ε_τ - quantifies the requirement of $|T_o(j\omega)|$ to be small.
- ν - the required roll-off

As before, we introduce stable transfer function $W_\tau(s)$ such that

$$|W_\tau(i\omega)| = \begin{cases} \dfrac{1}{1+\delta_\tau}, & \forall \omega \leq \omega_1 \\ 1/\varepsilon_\tau \left(\omega/\omega_1\right)^\nu, & \forall \omega > \omega_1 \end{cases}$$

The problem can now be formulated as finding a stabilizing controller guaranteeing

$$\|W_s(s)S_o(s)\|_\infty < 1, \quad \|W_\tau(s)T_o(s)\|_\infty < 1$$

This problem is not readily solvable, but can be approximated by:

$$\left\| \begin{bmatrix} W_\delta(s)\, S_o(s) \\ W_\tau(s)\, T_o(s) \end{bmatrix} \right\|_\infty < 1 \qquad (2)$$

(The latter formulation is restrictive, see prob. 2 in exercise 4.)

Sometimes, $C S_o$ is penalized instead of T_o, i.e.,

$$\left\| \begin{bmatrix} W_\delta(s)\, S_o(s) \\ W_x(s)\, C(s)\, S_o(s) \end{bmatrix} \right\|_\infty < 1 \quad , \qquad (3)$$

where $W_x(s)$ is chosen so that

$$|W_x(i\omega)| < \begin{cases} \dfrac{1}{1+\delta_x} \, , & \forall \omega \leq \omega_1 \\ 1/\varepsilon_x \left(\omega/\omega_1\right)^\nu , & \forall \omega > \omega_1 \end{cases}$$

* In SISO case $C \cdot S = T/P$

* Typically, using CS_o instead of T enables to decrease the order of the weight.

Optimizations (2) and (3) are usually called <u>mixed sensitivity problems.</u>

The optimization in (3) is a special case of the standard problem with

$$G = \left[\begin{array}{c|c} W_\sigma & -W_\sigma P \\ 0 & W_x \\ \hline 1 & -P \end{array}\right]$$

Some remarks:

- The discussion above provides guidlines for choosing the weights.
- The reasoning above is applicable in both SISO and MIMO settings.
- The formulation can further be extended to account for d and n inputs, i.e., for

$$G = \left[\begin{array}{ccc|c} W_\sigma & -W_\sigma P W_d & 0 & -W_\sigma P \\ 0 & 0 & 0 & W_x \\ \hline I & -P W_d & -W_n & P \end{array}\right]$$

<u>What did we study today?</u>

- Explicit state-space solutions for H_2 and H_∞ problems
- Interpretation for these standard problems
- Ideas for how to choose the weights.