Ph.D. course on Network Dynamics

Homework 5

To be discussed on Wednesday, January 25, 2012

In class, we have considered network dynamics of the following form. Given a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a finite local state (or 'opinion') space $\mathcal{X}$, and local transition kernels $\Psi_v(x'|x_v, x_{\mathcal{N}_v})$ for every $v \in \mathcal{V}$, we consider a stochastic process (in fact, a continuous-time Markov chain) $X(t) = \{X_v(t) : v \in \mathcal{V}\}$ on the product space $\mathcal{X}^{\mathcal{V}}$, whereby every node (often called 'agent') $v \in \mathcal{V}$ gets activated at the ticking of an independent rate-1 Poisson clock, and, if activated at some time $t \geq 0$, it updates its opinion to $X_v(t^+) = x'$ with conditional probability $\Psi_v\left(x'|X_v(t), X_{\mathcal{N}_v}(t)\right)$.

We have argued that, if the transition kernels are homogeneous, i.e., if $\Psi_v\left(x'|x_v, x_{\mathcal{N}_v}\right) = \tilde{\Psi}\left(x'|x_v, d_v, x_{\mathcal{N}_v}\right)$, and if the population is totally mixed (i.e., the graph $\mathcal{G}$ is the complete one on n nodes), then one can give up on keeping track of the opinions of the single agents and rather take a 'Eulerian' viewpoint, and look at the evolution of the empirical density of the agents' opinions, defined by $\rho^n_x := n^{-1}|\{v \in \mathcal{V} : X_v(t) = x\}|$. In fact, under such assumptions of total mixing of the population and homogeneity of agents' behavior, the vector $\rho^n(t) := \{\rho^n_x(t) : x \in \mathcal{X}\}$ forms a Markov chain on the space of opinion types $\mathcal{P}_n(\mathcal{X}) := \{\rho \in \mathbb{R}^{\mathcal{X}} : n\rho_x = n_x \in \mathbb{Z}_+, \sum_x n_x = n\}$.

Moreover, Kurtz's theorem guarantees that, under the additional technical assumption that $\tilde{\Psi}(x'|x, \rho)$ is Lipschitz-continuous in ρ for every $x, x' \in \mathcal{X}$, there exist positive constants K, K' such that, for all $T > 0$, $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} ||\rho^n(t) - \rho(t)|| \geq \varepsilon\right) \leq K' \exp(-Kn\varepsilon^2/T)\,,$$

where $\rho(t)$ is the solution of the Cauchy problem associated to the mean-field

ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_x = \sum_y \rho_y \tilde{\Psi}(x|y,\rho) - \rho_x \,, \qquad x \in \mathcal{X} \,,$$

with initial value $\rho(0) = \rho^n(0)$.

Exercise 1. *Prove that $|\mathcal{P}_n(\mathcal{X})| = \binom{n+|\mathcal{X}|-1}{|\mathcal{X}|-1}$. (Hint: one has to count the different ways of assigning n identical balls to $|\mathcal{X}|$ distinguished urns. Consider $|\mathcal{X}|-1$ identical bars whose role is just separating the balls belonging to the different $|\mathcal{X}|$ urns. Then, one is left with counting the number of possible ways of placing such separating objects in a universe of $n+|\mathcal{X}|-1$ elements...) Contrast the polynomial growth rate of $\binom{n+|\mathcal{X}|-1}{|\mathcal{X}|-1}$ with the exponential one of $\mathcal{X}^{\mathcal{V}}$ as $|\mathcal{V}|$ grows large with $|\mathcal{X}|$ constant.*

Exercise 2 (mean-field limit of the (smoothed) majority rule dynamics)**.** *The (smoothed) majority rule dynamics is characterized by local state space $\mathcal{X} = \{0, 1\}$ and transition kernel $\Psi_v(1|x_v, x_{\mathcal{N}_v}) = \Phi(d_v^{-1} \sum_{w\in\mathcal{N}_v} x_w)$, where $\Phi : [0, 1] \to [0, 1]$ is Lipschitz continuous and such that $\Phi(0) = 0$, and $\Phi(1 - x) = 1 - \Phi(x)$ for all $x \in [0, 1]$.*

(a) *Prove that the mean-field limit ODE is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_1 = \Phi(\rho_1) - \rho_1 \,; \tag{1}$$

(b) *Find the equilibria of (1), discuss their stability and characterize their basin of attraction.*

Exercise 3 (k-majority rule dynamics)**.** *Let k be a positive odd integer and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a graph such that $d_v \geq k$ for all $v \in \mathcal{V}$. Consider the k-majority rule dynamics whereby each agent, when activated, selects k distinct neighbors of its uniformly at random and moves towards the opinion held by the majority of them. Find the mean-field ODE characterizing the limit when $N \to +\infty$ and study the asymptotic behavior (as t grows large) of the associated initial value problem.*

Exercise 4 (mean-field limit of the SIS epidemics)**.** *Consider the SIS epidemics with rate-$\frac{\gamma}{n-1}$ transmission time on every link with one infected and one susceptible end-node, and rate-$(1-\gamma)$ exponential recovery time, where $\gamma \in (0, 1)$.*

(a) *Define a local state space $\mathcal{X}$ and write down the transition kernel $\Psi_v(x'|x, x_{\mathcal{N}_v})$ for every $v \in \mathcal{V}$ (hint: assume that a node gets activated whenever it has a potential recovery or gets a potential decease transmission from one of the other $n-1$ nodes)*

(b) *What are all the absorbing states in $\mathcal{X}^{\mathcal{V}}$ of the Markov chain $X(t)$ on a general connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$?*

(c) *Prove that the mean-field limit ODE is*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\rho_S &=& (1-2\gamma)\rho_I + \gamma\rho_I^2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho_I &=& -(1-2\gamma)\rho_I - \gamma\rho_I^2 \end{cases} \tag{2}$$

(d) *What are the equilibria of (3)? Discuss their stability as γ varies in $(0, 1)$.*

Exercise 5 (mean-field limit of the SIR epidemics)**.** *Consider the SIR epidemics with rate-$\frac{\gamma}{n-1}$ transmission time on every link with one infected and one susceptible end-node, and rate-$(1-\gamma)$ exponential recovery time, where $\gamma \in (0, 1)$.*

(a) *Define a local state space $\mathcal{X}$ and write down the transition kernel $\Psi_v(x'|x, x_{\mathcal{N}_v})$ for every $v \in \mathcal{V}$ (hint: assume that a node gets activated whenever it has a potential recovery or gets a potential decease transmission from one of the other $n-1$ nodes)*

(b) *What are all the absorbing states in $\mathcal{X}^{\mathcal{V}}$ of the Markov chain $X(t)$ on a general connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$? And what are they on the complete graph?*

(c) *Prove that the mean-field limit ODE is*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\rho_S &=& -\gamma\rho_S\rho_I \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho_I &=& \gamma\rho_S\rho_I - (1-\gamma)\rho_I \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho_R &=& (1-\gamma)\rho_I \end{cases} \tag{3}$$

(d) *Plot the flow diagram of (3);*

(e) *What are the equilibria of (3)?*

(f) *Prove that every trajectory of (3) is convergent to some limit $\rho^* \in \mathcal{P}(\mathcal{X})$, which depends on the initial condition $\rho(0)$; (hint: $\rho_S(t)$ is non-increasing, $\rho_R(t)$ non-decreasing, and the trajestoris belong to the compact $\mathcal{P}(\mathcal{X})$)*

(g) *Let $\mathcal{R} := \{\rho \in \mathcal{P}(\mathcal{X}) : \rho_I = 0\}$, $\mathcal{R}^S_\gamma := \{\rho \in \mathcal{R} : \rho_S \leq 1/\gamma - 1\}$, and $\mathcal{R}^U_\gamma := \mathcal{R} \backslash \mathcal{R}^S_\gamma$. Prove that, for every initial condition $\rho(0) \in \mathcal{P}(\mathcal{X}) \backslash \mathcal{R}^U_\gamma$, the solution of (3) satisfies*

$$\lim_{t \to +\infty} \rho(t) \in \mathcal{R}^S_\gamma \,;$$

(h) *Conclude that, for $\gamma > 1/2$,*

$$\rho_I(0) > 0 \quad \Longrightarrow \quad \lim_{t \to \infty} \rho_R(t) \geq 2 - 1/\gamma > 0 \,.$$

Exercise 6. *Consider the following heterogeneous voter model. We have a totally mixed population consisting of N_α people of type α and N_β of type β. All people are selected with the same probability. When an α-agent (respectively, a β-agent) is activated, it chooses a neighbor uniformly at random in the entire population (who can be either an α or a β) and takes its opinion with probability q_α (resp. q_β).*

(a) *Write down the transition kernel;*

(b) *Propose a meaningful mean-field approach to this model and following the ideas discussed before Theorem 1 in the notes (discrete dynamics over networks), find the system of differential equations describing the limit when $N = N_\alpha + N_\beta \to +\infty$ under the assumption that $N_\alpha/N \to \nu_\alpha$ and $N_\beta/N \to \nu_\beta$.*

(c) *Study the asymptotic behavior of the system of differential equations found in item (a) as a function of the four parameters ν_α, ν_β, q_α, q_β.*

(d) *Discuss theoretically or numerically the effect of adding flipping terms in the dynamics accounting for spontaneous opinion flipping (as in the 'classical' voter model).*