Ph.D. course on Network Dynamics

Homework 2

To be discussed on Tuesday, November 23, 2011

Exercise 1. *Have a look at the notes on on the Erdös-Rényi branching process approximation and connectivity phase transition that are posted on the course webpage. Try to fill in the gaps of the proofs. Please, note there might be a few typos (e.g., the is a missing + before the last two summations in the big displayed equation at the top of page 2); please be patient until I find the .tex file.*

The following exercises deal with the fixed degree distribution random graphs defined via the configuration model. Let us recall a bit of notation. We start with a (node-perspective) degree distribution $\{p_k : k \geq 0\}$, and let $\mu := \sum_k k p_k$ be its first moment, which will always be assumed finite. We also let

$$q_k := \frac{1}{\mu}(k+1)p_{k+1}\,, \quad k \geq 0\,, \qquad \nu := \sum_k q_k k = \frac{1}{\mu}\sum_k p_k k(k-1)\,,$$

be the edge-perspective degree distribution and its first moment.

Exercise 2. *Let $p_0 = 0$ and, for $k \geq 1$, $p_k = C_\beta k^{-\beta}$ with $\beta > 1$, and $C_\beta := \left(\sum_{k\geq 1} k^{-\beta}\right)^{-1}$.*

(a) *prove that μ is finite if and only if $\beta > 2$, while ν is finite if and only if $\beta > 3$;*

Let $p_k = e^{-\lambda}\lambda^k/k!$, for $k \geq 0$;

(b) *prove that $q_k = p_k$, for $k \geq 0$.*

Conversely, assume that $q_k = p_k$, for $k \geq 0$, and

(c) *prove that $p_k = e^{-\lambda}\lambda^k/k!$, for $k \geq 0$, where $\lambda = \sum_k kp_k = \mu$.*

I.e., the Poisson distribution is the unique distribution which is the same from node and edge perspective.

Let us briefly recall the definition of the configuration model, which was somehow misrepresented in the first attempt in class. For a given node size $n \geq 1$, we generate n i.i.d. random variables, $d_1, d_2, \ldots, d_n$, such that $\mathbb{P}(d_v = k) = p_k$ for all $v = 1, \ldots, n$, and $k \geq 0$, we define $\Delta := \sum_{v=1}^n d_v$, and we condition on the event $\{\Delta \text{ is even}\}$. Finally, we generate the graph with node set $\mathcal{V} = \{1, 2, \ldots, n\}$ by matching the Δ 'half-edges' attached to the n nodes uniformly at random. I.e., we iteratively form edges by first choosing a pair of half-edges uniformly at random from the $\binom{\Delta}{2}$ possibilities and matching them; then choosing another pair of half-edges uniformly at random from the remaining $\binom{\Delta-2}{2}$ possibilities and matching them; and so on, until running out of half-edges (which will inevitably happen after $\Delta/2$ steps since we have conditioned on Δ being even). Alternatively, draw the n nodes with their Δ half-edges on the left and draw other $\Delta/2$ 'virtual nodes' each with two half-edges; then, match the Δ half-edges on the left with those on the right by a random uniform permutation; finally, make the $\Delta/2$ virtual nodes of degree 2 'disappear' by letting their two neighbors (which are necessarily 'true' nodes) be directly connected by an edge.

Exercise 3. *As we discussed, this configuration model possibly leads to self-loops and parallel edges. Let χ_1 be the number of self-loops and χ_2 be the number of self-loops. In class, we proved that $\mathbb{E}[\chi_1]$ converges to $\nu/2$ as n grows large. Prove that $\mathbb{E}[\chi_2]$ converges to $\nu^2/4$ as n grows large. (In fact, assuming finite ν, Theorem 3.1.2 of Durrett shows that χ_1 and χ_2 are asymptotically independent with marginals Poisson($\nu/2$) and Poisson($\nu^2/4$), respectively).*

Exercise 4. *Assume $\mu < +\infty$, and $\nu < +\infty$. Consider the two-stage branching process with $Z_0 = 1$, and $Z_{t+1} = \sum_{i=1}^{Z_t} X_i^{t+1}$, where, X_i^t are independent random variables, with $\mathbb{P}(X_1^1 = k) = p_k$, and $\mathbb{P}(X_i^t = k) = q_k$, for all $k \geq 0$, $i \geq 1$, and $t \geq 2$. Let $\rho_t := \mathbb{P}(Z_t = 0)$, and $\rho_{ext} := \lim_{t\to+\infty} \rho_t$. Prove that*

(a) *if $\nu < 1$, $\rho_{ext} = 1$;*

(b) *if* $\nu > 1$, $\rho_{ext} = 1 - \Phi(1 - \overline{\rho}_{ext}) \in [0, 1)$, *where* $\Phi(y) = \sum_k p_k z^k$ *is the generating function of the first offspring generation, and* $\overline{\rho}$ *is the smallest fixed point in* $[0, 1]$ *of* $\overline{\Phi}(y) := \sum_k q_k y^k$, *i.e., the generating function of the offspring generations after the first one.*