# Network Dynamics: an Overview

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# This class

- 15 two-hours meetings (including today)
- $\sim$ 1 lecture + 1 exercise session per week
- 9 ECTS
- homework's + small final presentation
- reference books:
  1. Newman, 'Networks', Oxford U.P., 2010 (general intro)
  2. Vega-Redondo, 'Complex social networks', Cambridge U.P., 2007 (general intro)
  3. Durrett, 'Random graph dynamics', Cambridge U.P., 2007 (mathematics, advanced)
  4. draft of a new book (mathematics, undergrad) http://www.win.tue.nl/~rhofstad/NotesRGCN.html
  5. ...
- today: course overview

# Similar courses around the world

- MIT grad (by J.Tsistsiklis $ P.Jaillet): http://stellar.mit.edu/S/course/6/sp11/6.986/
- MIT undergrad (by D.Acemoglu & A.Ozdaglar): http://stellar.mit.edu/S/course/6/fa09/6.207J/
- Cornell undergrad (by D.Easley, J.Kleinberg, & E.Tardos):: http://www.infosci.cornell.edu/courses/info2040/2011fa/
- Berkeley grad (by E.Mossel): http://www.stat.berkeley.edu/ mossel/teach/SocialChoiceNetworks10/index.html
- North-Easter U. (by A.Barabasi): http://barabasilab.neu.edu/courses/phys5116/
- Michigan U. (by L.Adamic): http://open.umich.edu/education/si/si508/fall2008/materials

# (Complex) networks

(Large-scale) systems of (simple) interacting units

- infrastructure networks: transportation, power, Internet
- informational networks: WWW, citation networks
- social networks: friendships, family ties, Facebook etc.
- economic and financial networks: supply chains, borrowing-lending nets
- biological networks: neural networks, gene/protein interactions
- ecological networks: food webs, flocks, ...

# Studying (complex) networks

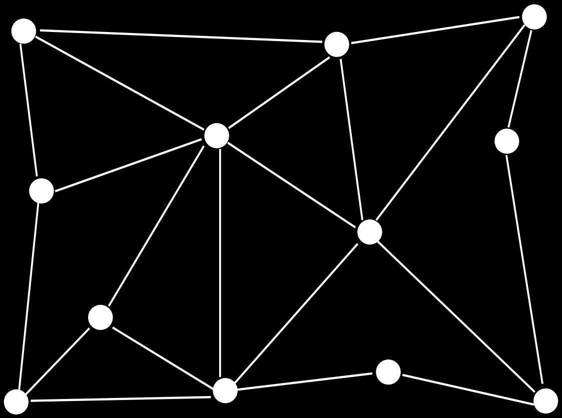
network structure + interaction mechanism

⇓

emerging behavior

- spread of epidemics and information
- design of (distributed) algorithms
- opinion formation, social influence and learning
- network robustness,
- cascaded failures, systemic risk
- ...

# Mathematical representation of network structure

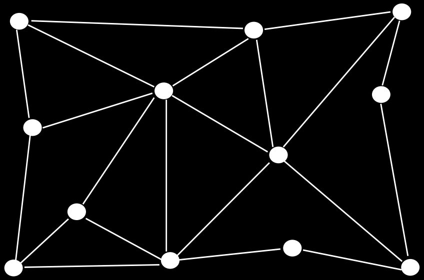


(un)directed (weighted) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$\mathcal{V}$ = set of vertices (or nodes) $\quad n = |\mathcal{V}| < +\infty$

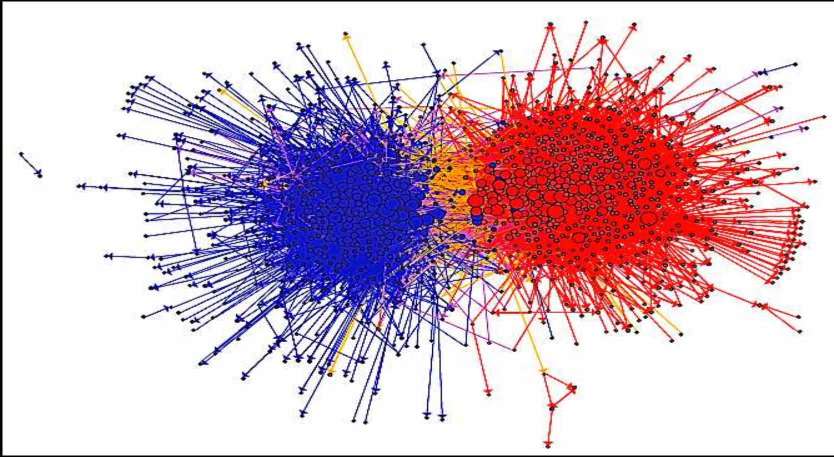
$\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} =$ set of edges (or links)

# Examples 1



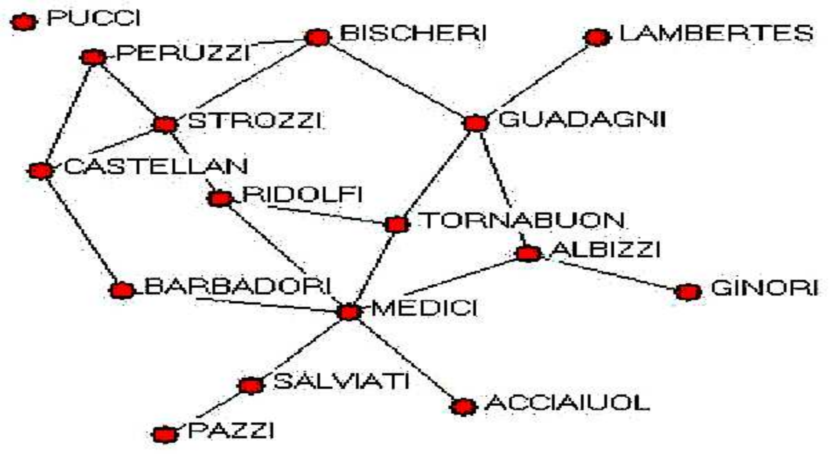
- Internet: nodes=routers, edges=direct physical links (und.)
- traffic networks: nodes=junctions, links=roads (directed)
- actors collaboration: nodes=actors, link⇔ same movie (und.)
- scientific collab.: nodes=researchers, link=coauthors (und.)

# Example 2: political blogs before 2004 US elections



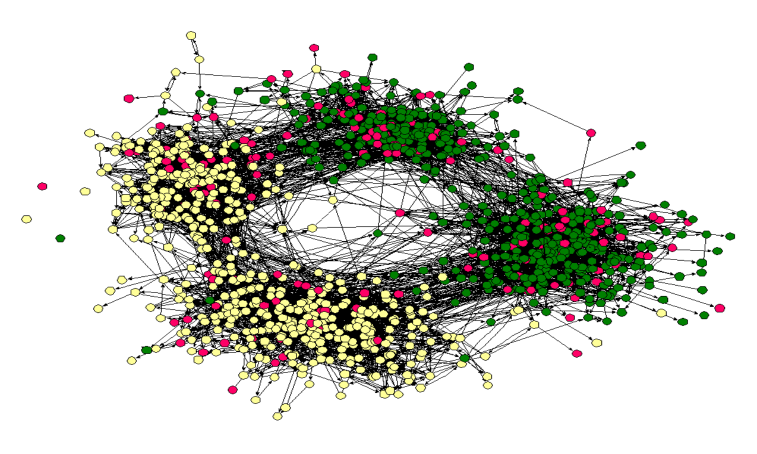
from Adamic and Glance, 'The Political Blogosphere and the 2004 U.S. Election: Divided They Blog', 2005

# Example 3: Family ties in 15th century Florence



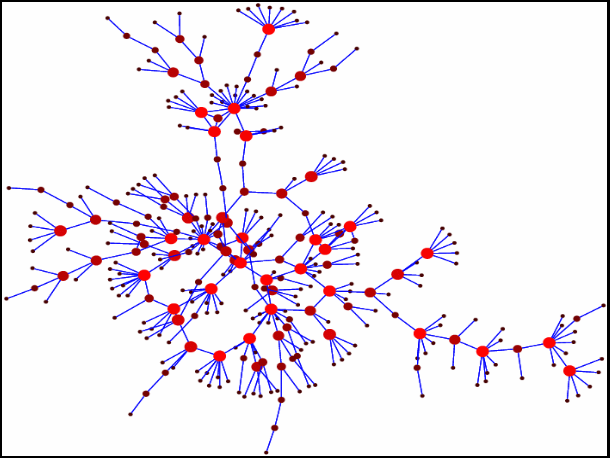

from Padgett and Ansell, ‘Robust action and the rise of the Medici, 1400-1434’, 1993

# Example 4: High school friendships



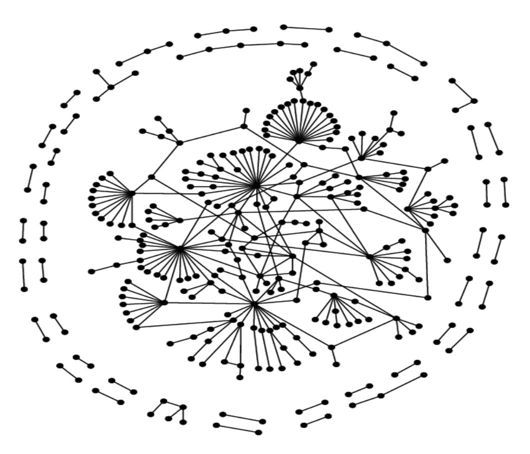
from Moody, 'Race, school integration, and friendship segregation in America', 2002

# Example 5: Sexual contacts



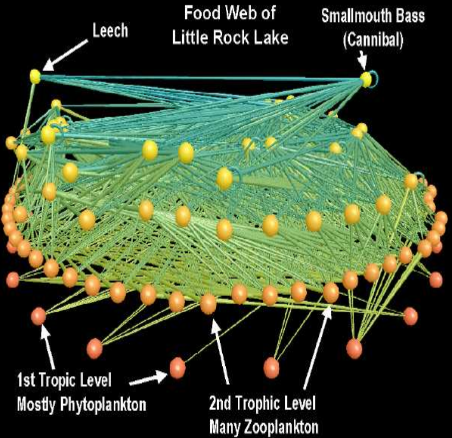
from Newman, ‘The structure and function of complex systems’, 2003

# Example 6: protein network in yeast nucleus



from Maslov and Sneppen ‘Specicity and stability in topology of protein networks’, 2002

# Example 7: Freshwater food web




from Martinez, 'Artifacts or attributes? Effects of resolution on the Little Rock Lake food web', 1991

# Random networks

network structure + interaction mechanism

$\Downarrow$

emerging behavior

structure of large-scale networks is difficult to describe exactly: huge or non directly accessible data

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network structure + interaction mechanism

$\Downarrow$

emerging behavior

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aggregate properties:

- connectivity, diameter / average distance
- frequency of subgraphs
- degree distribution

statistical approach:

- ensemble of graphs
- typical properties as $n = |\mathcal{V}| \to +\infty$

# Complex networks

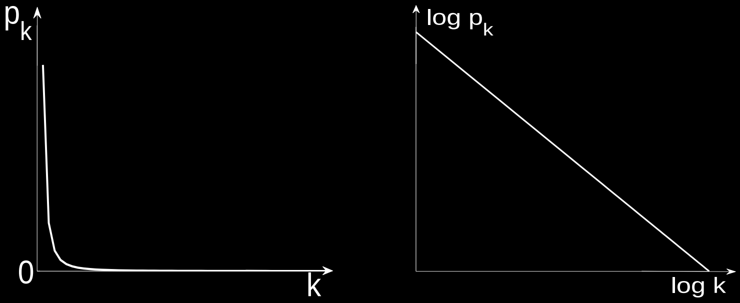
Properties widely observed in empirical studies

1. small world $\leftrightarrow$ diameter $\approx \log n$
2. high clustering $\leftrightarrow$ many triangles
3. scale free $\leftrightarrow$ power law degree distribution

# Small world

- Milgram's experiment ('67): randomly selected group of few hundreds of people from Omaha (NE). A letter given to each of them to be delivered to a stock broker living in Boston (MA). Letter can only be handed to a person know directly.
  $\Longrightarrow$ 35% letters reach destination, median # of steps: 5.5
  $\Longrightarrow$ "6 degrees of separation"
- Albert, Jeong, and Barabasi ('99): WWW network, $n \sim 800M$ average distance of webpages $\sim 0.35 + 2.06 \log n = 18.59$

# Power laws




$$p_k \sim C k^{-\gamma}$$

show up in quite different contexts:

- percentage of words in a book
- percentage of cities of a given size
- percentage of people having a certain income

Power law $\Longrightarrow$ heavy tails: lots of large cities, lots of rich people

Typically explained by rich-gets-richer mechanisms

# Power law networks

$$d_v := \#\{\text{neighbors of } v\} \qquad p_d := \frac{1}{n}\#\{v : d_v = d\}$$

$$p_d \sim C d^{-\gamma}$$

Empirical studies:

- Barabasi and Albert ('99): WWW has $\gamma_{in} \sim 2.1$, $\gamma_{out} \sim 2.7$
- Faloutos ('99): Internet $\gamma \sim 2.16$
- actor collaborations: $\gamma \sim 2.3$
- Redner ('98): citation network: $\gamma_{in} \sim 2.6$, $p_d^{out} \sim C\exp(-Kd)$
- Liljeros ('01): # sexual partners per year (in Sweden) $\gamma_{male} \sim 3.3$, $\gamma_{female} \sim 3.5$

$$2 < \gamma \leq 3 \quad \Longrightarrow \quad \langle d \rangle < +\infty \quad \langle d^2 \rangle = +\infty$$

$$\gamma \geq 3 \quad \Longrightarrow \quad \langle d \rangle < +\infty \quad \langle d^2 \rangle < +\infty$$

power law $\leftrightarrow$ scale free

# Random graphs 1: Erdös-Rényi

$$\mathcal{G}(n, p) = (\mathcal{V}, \mathcal{E}) \qquad |\mathcal{V}| = n$$

$$\mathbb{P}\left(\{v, w\} \in \mathcal{E}\right) = p \qquad \text{mutually independent}$$

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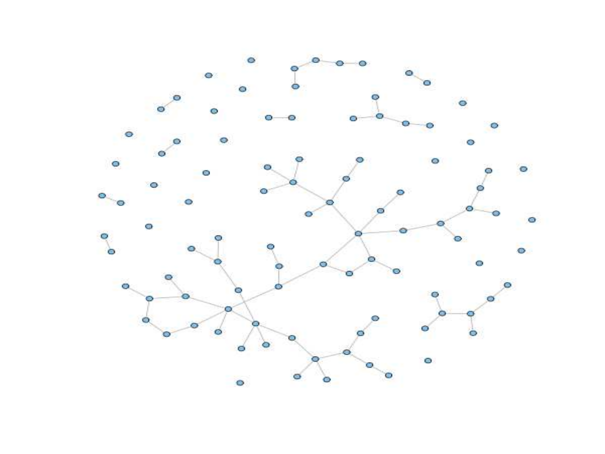

$n = 100$
$p = 0.15$

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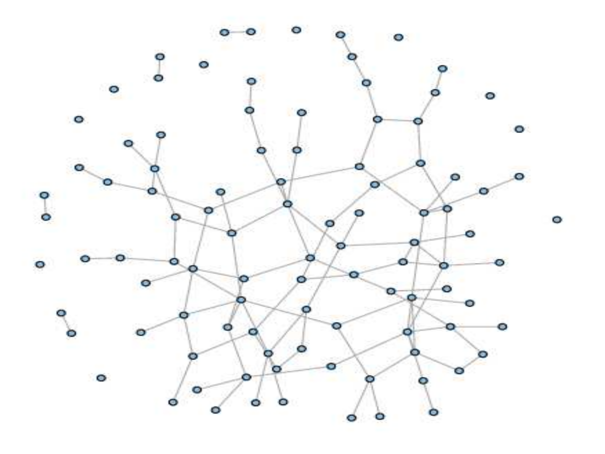
$n = 100$
$p = 0.2$



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If $p = \lambda/n$, with high probability as $n \to \infty$,

phase transition

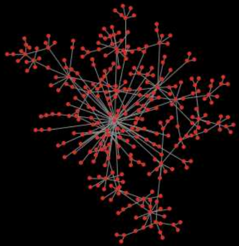
$$\begin{cases} \lambda < 1 \Longrightarrow \text{size(largest component)} \asymp \log n \\ \lambda > 1 \Longrightarrow \begin{array}{l} \text{size(largest component)} \asymp n \\ \text{diam(giant component)} \asymp \log n \end{array} \Bigg\} \text{ small world} \end{cases}$$

- Poisson degree distribution: $p_d \sim \dfrac{\lambda^d}{e^\lambda d!} \quad \Rightarrow \quad$ NO power law
- limited #(triangles) $\quad \Rightarrow \quad$ NO clustering

# Random graphs 2: preferential attachment

Barabasi-Albert ('99)

1. start from a small given graph $n_0$
2. add a vertex and connect it with $d$ older vertices randomly chosen with conditional probability $\propto$ their current degree
3. repeat step 2 $n - n_0$ times



http://ccl.northwestern.edu/netlogo/models/run.cgi?PreferentialAttachm

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$$\Longrightarrow \quad \begin{cases} p_d \sim Cd^{-3} \Rightarrow \text{power law} \\ diam \asymp \log n \Rightarrow \text{small world} \\ \text{(suitably modified)} \Rightarrow \text{high clustering} \end{cases}$$

# Dynamics over networks

network structure + interaction mechanism

$\Downarrow$

emerging behavior

- random walks
- contact processes / epidemics
- linear interactions: distributed averaging / voter model
- non-linear interactions: bounded confidence, majority model, evolutionary games

# Random walk

A particle moving from node to node, jumping at Poisson times to a randomly chosen neighbor of the current position

- $V(t) \in \mathcal{V}$ position of the particle at time $t$
- network connected $\implies$ unique stationary distribution $\pi_v = d_v / \sum_w d_w$
- how fast does $||\mathbb{P}(V(t) = \cdot) - \pi||$ go to 0 as a function of the network structure?
- when will $V(t)$ hit some other $w \in \mathcal{V}$?
- when will two independent copies of $V(t)$ started in different nodes meet for the first time?

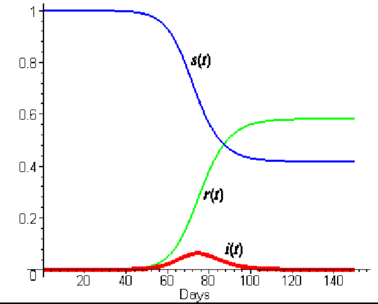
# Epidemics

SIR model (susceptible-infected-resistant)

- every node $v$ has a state $x_v(t) \in \{S, I, R\}$
- if a susceptible vertex has an infected neighbor, it becomes infected at rate $p$
- an infected vertex can spontaneously remove the infection and become resistant at rate $q$.

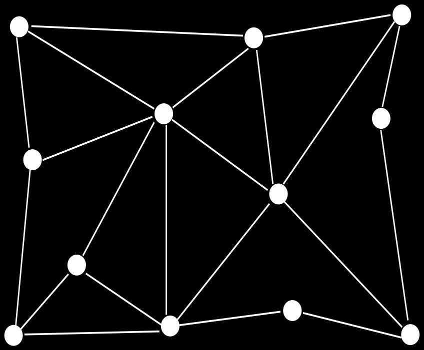
Starting from few infected nodes, what is the probability that infection will extend to a large part of the network?
How will this depend on $p$, $q$, and the network structure?

# Epidemics




- in power law network with $\gamma > 3$ for small enough $p/q$ an infection SIR will NOT spread, with high probability, to a large part of the network;
- in power law network with $2 < \gamma \leq 3$, an infection SIR will spread independently from the values of $p$ and $q$!
  The reason is the existence of high degree vertices.
  How many vertices do you need to immunize to stop the epidemics spreading?

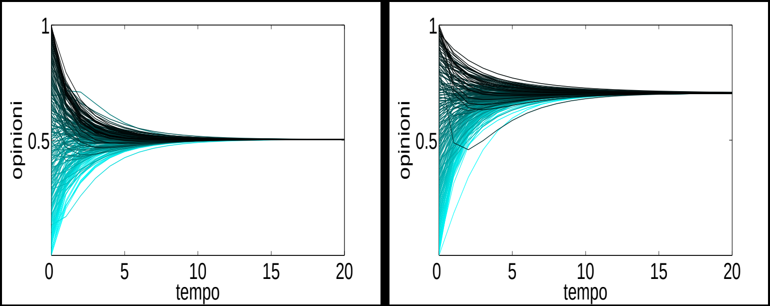
# Distributed averaging



Gossip model:

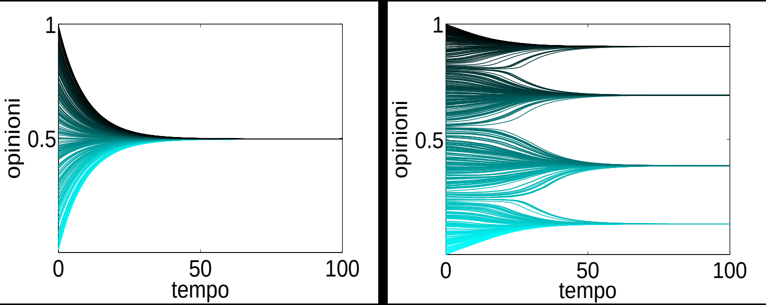
- every node $v$ has a state $x_v(t) \in \mathbb{R}$
- nodes get activated at independent Poisson times
- when a node $v$ is activated, it choses a neighbor $w$ at random and updates its value to $x_v(t) = (1-\omega)x_v(t^-) + \omega x_w(t^-)$
- network connected $\Longrightarrow$ convergence to consensus

# Distributed averaging




How is the limit consensus value related to the initial states of the nodes?
How does the network structure affect the speed of convergence?
What is the effect of heterogeneity of the agents behavior?

# Bounded confidence




- every node $v$ has a state $x_v(t) \in \mathbb{R}$
- like distribute averaging but updates only if $|x_x(t) - x_w(t)| < \theta$
- consensus vs fragmentation depending on confidence threshold $\theta$

# Next meeting

- basics of graph theory
- Wed 9-11-2011 15:00-17:00