Ph.D. course on Network Dynamics

Homework 1

To be discussed on Tuesday, October 8, 2013

Exercise 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the complete graph with n nodes, and P the stochastic matrix associated to the standard random walk on $\mathcal{G}$. Determine the spectral gap of P (i.e., $1 - \lambda_2$, where λ_2 is the second largest eigenvalue of P).*

Exercise 2. *Prove that all the entries of the invariant probability distribution π of an irreducible stochastic matrix P are strictly positive. (*hint: recall the argument used in class to prove uniqueness*)*

Exercise 3. *Consider the stochastic matrix*

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 \\ 2/3 & 0 & 0 & 0 & 1/3 \end{bmatrix}.$$

*Find its invariant probability distribution and its spectrum. Is P reversible? (*hint: *to compute eigenvectors, use the fact that an orthonormal basis of eigenvectors is $\{v^{(k)} : 0 \leq k \leq 4\}$ where $v^{(k)}_j = \frac{1}{\sqrt{5}} e^{i2\pi kj/5}$ for $1 \leq j \leq 5$, where i is the imaginary unit.)*

Exercise 4 (Birth and death chain)**.** *Consider a Markov chain over $\mathcal{V} = \{0, 1, \ldots, n\}$ with $P_{ij} = 0$ for all $|i - j| > 1$. Assume that $P_{ij} > 0$ whenever $|i-j| = 1$ (which makes P irreducible). Find the invariant probability distribution π and prove that P is reversible. (*hint: do both things at once, using reversibility to determine π_{k+1}/π_k, $k = 0, 1, \ldots, n-1$.*)*

Exercise 5 (Lazy random walk on the hypercube)**.** *The hypercube of dimension* $k \geq 1$ *is the graph with node set* $\mathcal{V} = \{0,1\}^k$ *and* $\mathcal{E} := \{\{u,v\} : ||u-v|| = 1\}$*. Let* P *be the stochastic matrix associated to the lazy random walk on* $\mathcal{G}$*. Determine an upper bound on the spectral gap* $1 - \lambda_2$ *using its variational characterization (*hint: take $\mathcal{S} = \{v \in \{0,1\}^k : v_1 = 0\}$ and f equal to 1 on $\mathcal{S}$ and to -1 on $\mathcal{V} \setminus \mathcal{S}$*)*

Exercise 6 (Lazy random walk on the cycle)**.** *The cycle of size* n *is the graph* $(\mathcal{V}, \mathcal{E})$ *with* $\mathcal{V} = \{0, \ldots, n-1\}$ *and* $\mathcal{E} := \{\{u,v\} : |u-v| = 1\}$ *where the difference is taken modulo* n *(so that* 0 *and* $n-1$ *are linked to each other). Use Cheeger's inequality to get a lower bound on the spectral gap.*