Ph.D. course on Network Dynamics

Homework 1

To be discussed on Tuesday, November 15, 2011

Exercise 1. *Prove the hand-shacking lemma: in every undirected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

$$2|\mathcal{E}| = \sum_{v \in \mathcal{V}} d_v \,,$$

where d_v *denotes the degree of a node* $v \in \mathcal{V}$.

Exercise 2. *Prove that every tree* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *(i.e., a connected undirected graph containing no cycles) satisfies*

$$|\mathcal{V}| = |\mathcal{E}| + 1 \,.$$

Hint: use induction on the number of nodes $\mathcal{V}$.

Exercise 3 (Properties of the Chernoff exponent)**.** *The moment generating function of a real-valued random variable* X *is defined as*

$$M_X(\theta) := \mathbb{E}\left[\exp(\theta X)\right] \in [0, +\infty] \,, \qquad \theta \in \mathbb{R} \,.$$

Observe that trivially $M_X(0) = 1$.

(a) *Prove that, if* $M_X(\theta^*) < +\infty$ *for some* $\overline{\theta} > 0$*, then* $M_X(\theta) < +\infty$ *for all* $\theta \in [0, \overline{\theta}]$*. Using the dominated convergence theorem, and the series expansion* $\exp(\theta X) = \sum_{k \geq 0} (\theta X)^k / k!$*, argue that*

$$M_X(\theta) = \sum_{k \geq 0} \frac{\theta^k}{k!} \mathbb{E}[X^k] \,, \qquad \forall \theta \in [0, \theta^*) \,,$$

where $\theta^* := \sup\{\theta :\ M_X(\theta) < +\infty\}$*. Conclude that, if* $\theta^* > 0$*, then*

$$\mathbb{E}[X^k] = \lim_{\theta \downarrow 0} \frac{\mathrm{d}^k}{\mathrm{d}\theta^k} M_X(\theta)\,, \qquad \forall k \geq 1\,,$$

which explains why $M_X(\theta)$ *is called the moment generating function.*

Now define the Chernoff exponent

$$h_X(a) := \sup\left\{\theta a - \log M_X(\theta) :\ \theta \geq 0\right\}\,, \qquad \forall a \in \mathbb{R}\,,$$

and prove that:

(b) $h_X(a) \geq 0$ *for all* $a \in \mathbb{R}$*; (this is easy!)*

(c) $h_X(a) = 0$ *for all* $a \leq \mathbb{E}[X]$*; (hint: apply Jensen's inequality* $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$ *to the convex function* $f(x) := \exp(x)$*, or to* $f(x) = -\log x$*)*

(d) *if* $M_X(\theta^*) < +\infty$ *for some* $\theta^* > 0$*, then* $h_X(a) > 0$ *for all* $a > \mathbb{E}[X]$*; (hint: compute the right derivative in* $\theta = 0$ *of* $f(\theta) = \theta a - \log M_X(\theta)$ *using point (a))*

(e) $h_X(a)$ *is non-decreasing in a; (easy, since it's defined as the sup of non-decreasing functions of a)*

(f) $h_X(a)$ *is convex in a; (also easy, since it's defined as the sup of linear functions of a)*

Exercise 4 (Chernoff exponent in special cases)**.** *Show that*

(a) *if* $X \sim$*Bernoulli*(p)*, then* $h_X(a) = a\log(a/p) + (1-a)\log((1-a)/(1-p))$*;*

(b) *if* $Y \sim$*Poisson*(λ)*, then* $h_Y(a) = a\log(a/\lambda) - a + \lambda$*.*

Prove the following useful estimates of the Chernoff exponent of a Bernoulli(p)*:*

(c) $h_X(a) \geq (a-p)^2/(2a)$*.*
(hint: use first order Taylor approximation with Lagrange residuals: $h_X(a) = h_X(p) + h'_X(p)(p-a) + h''_X(y)(p-a)^2/2$*, for some* $y \in [p, a]$*)*

Exercise 5. *Consider the Erdös-Rényi Ramdon graph $\mathcal{G}(n,p)$.*

(a) *Prove that, for all $\varepsilon \geq 0$,*

$$\mathbb{P}(d_v \geq (n-1)p(1+\varepsilon)) \leq \exp(-(n-1)p\varepsilon^2/(2(1+\varepsilon))) \qquad \forall v \in \{1,\dots,n\};$$

(hint: use Chernoff and Exercise 4(c))

(b) *let $d_{\max} := \max\{d_v : 1 \leq v \leq n\}$ be the maximum of the node degrees, and prove that, if $np \geq \lambda \log n$ where $\lambda > 1$, then*

$$\mathbb{P}(d_{\max} \geq 4pn) \overset{n\to+\infty}{\longrightarrow} 0\,.$$

(hint: use point (a) and the union bound)

(c) *prove that, for all $\varepsilon \geq 0$,*

$$\mathbb{P}(d_v \leq (n-1)p(1-\varepsilon)) \leq \exp(-(n-1)p\varepsilon^2/2) \qquad \forall v \in \{1,\dots,n\};$$

(hint: use Chernoff for $n-1-d_v$ which is Binomial$(n-1,(1-p))$ and argue as in Exercise 4(c) to get $a\log(a/p)+(1-a)\log((1-a)/(1-p)) \geq (p-a)^2/(2p)$)

(d) *let $d_{\min} := \min\{d_v : 1 \leq v \leq n\}$ be the maximum of the node degrees, and prove that, if $np \geq \lambda \log n$ where $\lambda > 2$, then there exists $\alpha(\lambda > 0)$ such that*

$$\mathbb{P}(d_{\min} \leq \alpha(\lambda)pn) \overset{n\to+\infty}{\longrightarrow} 0\,.$$

(hint: use point (c), and the union bound, and see that the argument works for every $\alpha \in (0, 1-\sqrt{2/\lambda})$)

Remark 1. *Durrett's Lemma 6.5.2 claims that our point (d) is true provided that only $\lambda > 1$ (instead of $\lambda > 2$, as we have assumed: his proof seems wrong to me, what are your thoughts?)*

Exercise 6. *Consider the Erdös-Rényi Ramdon graph $\mathcal{G}(n,p)$. For $v \in \{1,\dots,n\}$, and $k \geq 3$, let $N_k(v)$ be the number of cycles of length k passing through node v in $\mathcal{G}(n,p)$.*

(a) *Prove that*

$$\mathbb{E}[N_k(v)] = \frac{1}{2}(n-1)(n-2)\dots(n-k+1)p^k\,;$$

(Hint: show that the possible cycles containing v are $(n-1)(n-2)\dots(n-k+1)/2$, since one has to choose $k-1$ out of $n-1$ other nodes (beyond v) ...)

(b) *Using Markov's inequality, prove that*

$$\mathbb{P}(\exists \textit{cycle of length } \leq k \textit{ containing } v) \leq \begin{cases} \frac{1}{n}\frac{\lambda^3}{2}\frac{\lambda^{k-2}-1}{(\lambda-1)} & \textit{if } \lambda \neq 1 \\ \frac{1}{n}\frac{k-2}{2} & \textit{if } \lambda = 1 \end{cases}$$

Conclude that:

(c) *if $\lambda < 1$, then*

$$\mathbb{P}(\exists \textit{cycle containing } v) \leq \frac{\lambda^3 n^{-1}}{2(1-\lambda)} \overset{n\to+\infty}{\longrightarrow} 0\,;$$

(d) *if $\lambda > 1$, then*

$$\mathbb{P}(\exists \textit{cycle of length } \leq a\log n \textit{ containing } v) \leq \frac{\lambda n^{a\log\lambda - 1}}{2(\lambda-1)} \overset{n\to+\infty}{\longrightarrow} 0\,,$$

for all $a < 1/\log\lambda$

Exercise 7 (Supercritical branching process)**.** *Consider a branching process Z_t with offspring distribution $p_k := \mathbb{P}(X = k)$, let $\mu := \mathbb{E}[X] = \sum_k kp_k$ be the expected number of offsprings and $\Phi(y) := \mathbb{E}[y^X] = \sum_k p_k y^k$ the generating function of X. Assume that $\mu > 1$, and $p_0 < 1$, so that that the extinction probability ρ_{ext} is the unique solution in $(0,1)$ of $y = \Phi(y)$. Prove that*

(a) *the process conditioned on extinction, $\tilde{Z}_t$, is a branching process with offspring distribution having generating function*

$$\tilde{\Phi}(y) = \frac{\Phi(\rho_{ext}y)}{\rho_{ext}}\,;$$

(hint: if $\tilde{X}_1^1$ is the number of first generation offsprings with a finite line of descent, then $\mathbb{P}(\tilde{X}_1^1 = k,\, ext) = p_k\rho_{ext}^k$, for $k \geq 0$)

(b) *conditioned on survival, if one looks only at individuals that have an infinite line of descent, then one obtains a new branching process $\tilde{Z}_t$ with offspring distribution having generating function*

$$\tilde{\Phi}(y) = \frac{\Phi((1-\rho_{ext})y + \rho_{ext}) - \rho_{ext}}{1-\rho_{ext}}\,.$$

(hint: if $\tilde{X}_1^1$ is the number of first generation offsprings with an infinite line of descent, then $\mathbb{P}(\tilde{X}_1^1 = k) = \sum_{j\geq k} p_j \binom{j}{k}(1-\rho_{ext})^k\rho_{ext}^{j-k}$, for $k \geq 1$)

Exercise 8 (Subcritical branching process and Erdös-Rényi random graph)**.** *Consider a branching process* $Z_0 = 1$, $Z_{t+1} = \sum_{i=1}^{Z_t} X_i^t$ *with offspring distribution* $X_i^t \sim$*Binomial*(n,p). *Assume that* $\lambda = \mathbb{E}[X_i^t] = np < 1$.

(a) *Prove that the total size* $T := \sum_{t\geq 0} Z_t$ *satisfies*

$$\mathbb{P}(T \geq k) \leq \exp\left(-k\left(\lambda - 1 - \log\lambda\right)\right) ;$$

(hint: use Chernoff bound, the explicit computation of Exercise 4(a), and the inequality $\log(1+x) \leq x$*)*

(b) *conclude that, for all* $a \geq 0$

$$\mathbb{P}(T \geq a\log n) \leq n^{-a(\lambda-1-\log\lambda)} .$$

Now, let us consider the subcritical Erdös-Rényi random graph $\mathcal{G}(n,p)$ *with* $\lambda = pn < 1$. *Recall the epidemics interpretation for finding the size of the connected component of some node* $v \in \mathcal{V} := \{1,\ldots,n\}$:

$$\mathcal{S}_0 = \mathcal{V} \setminus \{v\} , \quad \mathcal{I}_0 = \{v\} , \quad \mathcal{R}_0 := \emptyset ,$$

$$\mathcal{S}_{t+1} = \mathcal{S}_t \backslash \mathcal{I}_{t+1} , \quad \mathcal{I}_{t+1} = \{j \in \mathcal{S}_t : \chi_{ij} = 1 \text{ for some } j \in \mathcal{I}_t\} , \quad \mathcal{R}_{t+1} := \mathcal{R}_t \cup \mathcal{I}_t .$$

Assume that (this was mentioned in the last class and will be proven in the next class), for every $v \in \mathcal{V}$ *one can construct a branching process* Z_t^v *with offspring distribution Binomial*(n,p) *such that*

$$|\mathcal{I}_t| \leq Z_t , \qquad \forall t \geq 0 .$$

Using point (b) and a union bound,

(c) *prove that, for every* $a > (\lambda - 1 - \log\lambda)^{-1}$

$$\mathbb{P}\left(\max_v |\mathcal{C}(v)| \geq a\log n\right) \overset{n\to+\infty}{\longrightarrow} 0 ,$$

i.e., the size of the largest component in subcritical $\mathcal{G}(n,\lambda/n)$ *is bounded from above by* $(\lambda - 1 - \log\lambda)^{-1}\log n$.