Robust Control 2000

Lecture 7

- RS and H_∞ Optimization of Coprime Factors.
- H_∞ Loop Shaping Procedure.
- Justification of H_∞ Loop Shaping.

Robust Stabilization of Coprime Factors

Left coprime factor uncertainty model:

$$P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N).$$

By Small Gain Theorem:

RS for $\|[\tilde{\Delta}_N \ \Delta_M]\|_\infty \le \epsilon$ iff

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \frac{1}{\epsilon}.$$

This is H_∞ optimization.

In the standard lower LFT form

$$\begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} = \mathcal{F}_l(G, K)$$

where

$$G = \left(\begin{array}{c|c} 0 & I \\ \tilde{M}^{-1} & -P \\ \hline \tilde{M}^{-1} & -P \end{array} \right).$$

State Space Formulas

Consider a state space representation of the strictly proper plant

$$P = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right).$$

It is easy to verify that

$$[\tilde{N} \ \tilde{M}] = \left(\begin{array}{c|cc} A + LC & B & L \\ \hline C & 0 & I \end{array} \right),$$

where $A + LC$ is stable, gives a left coprime factorization. Then

$$G = \left(\begin{array}{c|cc} A & -L & B \\ \hline \begin{pmatrix} 0 \\ C \end{pmatrix} & \begin{pmatrix} 0 \\ I \end{pmatrix} & \begin{pmatrix} I \\ 0 \end{pmatrix} \\ C & I & 0 \end{array} \right).$$

Note: $\boxed{D_{11} \neq 0.}$ Hence

$$\frac{1}{\epsilon_{opt}} = \gamma_{opt} > \|I\| = 1 \quad \Leftrightarrow \quad \epsilon_{opt} < 1.$$

H_∞ optimization of Coprime Factors

Apply H_∞ optimization result to G ([Zhou,Th. 14.7]). Two Hamiltonian matrices are

$$H_\infty = \begin{pmatrix} A - \frac{1}{\gamma^2-1}LC & \frac{1}{\gamma^2-1}LL^* - BB^* \\ -\frac{\gamma^2}{\gamma^2-1}C^*C & -(A - \frac{1}{\gamma^2-1}LC)^* \end{pmatrix},$$

$$J_\infty = \begin{pmatrix} (A + LC)^* & -C^*C \\ 0 & -(A + LC) \end{pmatrix}.$$

Note that $Y_\infty = 0$. Thus the result becomes

Theorem: Let $D = 0$. Then there exists a stabilizing controller K such that

$$\|\mathcal{F}_l(G, K)\|_\infty < \gamma$$

if and only if $\gamma > 1$, $H_\infty \in \mathrm{dom(Ric)}$ and $X_\infty = \mathrm{Ric}(H_\infty) \ge 0$.

Remark: The result depends on the choice of L, i.e. choice of coprime factors.

Normalized Coprime Factors

Let choose L such that $\tilde{M}$ and $\tilde{N}$ become the normalized left coprime factors.

Let Y be the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0.$$

The matrix $A - YC^*C$ is stable, so we can put

$$L = -YC^*.$$

Lemma: With the choice $L = -YC^*$ the left coprime factors become normalized.

Proof: Denote $\mathcal{A}(s) = (sI - A + YC^*C)^{-1}$ and calculate

$$\begin{aligned}
\begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} \tilde{N}^* \\ \tilde{M}^* \end{pmatrix} &= I - C\mathcal{A}YC^* - CY\mathcal{A}^*C^* + \\
&\quad + C\mathcal{A}(B^*B + YC^*CY)\mathcal{A}^*C^* = \\
&= I + C\mathcal{A}(B^*B + YC^*CY - Y(\mathcal{A}^*)^{-1} - \mathcal{A}^{-1}Y)\mathcal{A}^*C^* = \\
&= I + C\mathcal{A}(\underbrace{B^*B - YC^*CY + AY + YA^*}_{=0})\mathcal{A}^*C^* = I.
\end{aligned}$$

H_∞ Optimization of Normalized Coprime Factors

Theorem: Let $D = 0$ and $L = -YC^*$ where $Y \geq 0$ is the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0.$$

Then $P = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization and

$$\begin{aligned}
&\inf_{K-\text{stab}} \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \\
&= \frac{1}{\sqrt{1 - \lambda_{max}(YQ)}} = \left(1 - \|\tilde{N} \ \ \tilde{M}\|_H^2\right)^{-1/2} = \gamma_{opt}
\end{aligned}$$

where

$$Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0.$$

Moreover, a controller achieving $\gamma > \gamma_{opt}$ is

$$\begin{aligned}
K(s) &= \left(\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array} \right), \\
X_\infty &= \frac{\gamma^2}{\gamma^2 - 1} Q \left(I - \frac{\gamma^2}{\gamma^2 - 1} YQ \right)^{-1}.
\end{aligned}$$

Proof: Denote

$$H_q = \begin{pmatrix} A - YC^*C & 0 \\ -C^*C & -(A - YC^*C)^* \end{pmatrix},$$

$$T = \begin{pmatrix} I & -\frac{\gamma^2}{\gamma^2-1}Y \\ 0 & \frac{\gamma^2}{\gamma^2-1}I \end{pmatrix}.$$

It is straightforward to see that

$$H_\infty = TH_qT^{-1}.$$

Since $Q = \mathrm{Ric}(H_q)$ we have the stable invariant subspace for H_∞ as

$$T \begin{pmatrix} I \\ Q \end{pmatrix} = \begin{pmatrix} I - \frac{\gamma^2}{\gamma^2-1}YQ \\ \frac{\gamma^2}{\gamma^2-1}Q \end{pmatrix}.$$

Finally $\exists X_\infty \geq 0$ iff

$$I - \frac{\gamma^2}{\gamma^2 - 1}YQ > 0 \quad \Leftrightarrow \quad \gamma^2 > \frac{1}{1 - \lambda_{max}(YQ)}. \ \blacksquare$$

Note that Y and Q are controllability and observability Gramians for $[\tilde{N} \ \tilde{M}]$. Hence $\lambda_{max}(YQ)$ is the square of the Hankel norm of it.

Some related H_∞ problem

Since $\tilde{M}$, $\tilde{N}$ are the normalized lcf we have

$$\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} \begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix}^* = I.$$

Therefore

$$\begin{aligned}
&\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\| = \\
&\quad = \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} \begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} \right\| = \\
&\quad = \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1} \begin{pmatrix} I & P \end{pmatrix} \right\|.
\end{aligned}$$

Does not depend on factorization.

Corollary:

$$\begin{aligned}
&\inf_{K-\text{stab}} \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I + PK)^{-1} \begin{pmatrix} I & P \end{pmatrix} \right\|_\infty = \\
&= \frac{1}{\sqrt{1 - \lambda_{max}(YQ)}} = \left(1 - \|\tilde{N} \ \ \tilde{M}\|_H^2\right)^{-1/2} = \gamma_{opt}.
\end{aligned}$$

Right Coprime Factors

What if we have normalized rcf $P = NM^{-1}$?

Theorem:

$$\left\| \begin{pmatrix} I \\ K \end{pmatrix} (I+PK)^{-1} \begin{pmatrix} I & P \end{pmatrix} \right\| = \\ = \left\| \begin{pmatrix} I \\ P \end{pmatrix} (I+KP)^{-1} \begin{pmatrix} I & K \end{pmatrix} \right\|.$$

Corollary: Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the normalized rcf and lcf, respectively. Then

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \\ = \left\| M^{-1}(I+KP)^{-1} \begin{pmatrix} I & K \end{pmatrix} \right\|_\infty .$$

Conclusion: It does not matter what kind of factorization we have. One can work with either.

Stability Margin

Introduce a quantity $b_{P,K}$

$$b_{P,K} = \begin{cases} \left(\left\| \begin{pmatrix} I \\ K \end{pmatrix} (I+PK)^{-1} \begin{pmatrix} I & P \end{pmatrix} \right\|_\infty \right)^{-1} & \text{if } K \text{ stabilizes } P, \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{opt} = \sup_K b_{P,K}.$$

Then $b_{P,K} = b_{K,P}$ and

$$b_{opt} = \sqrt{1 - \lambda_{max}(YQ)} = \sqrt{1 - \| \tilde{N} \ \tilde{M} \|_H^2}.$$

It holds $0 \leq b_{opt} \leq 1$. The larger b_{opt} the more robustly stable the closed loop system.

This quantity is related to the classical gain and phase margins. Thus it can be considered as a general stability margin (Vinnicombe, 1993).

Relation to Gain and Phase Margins

Theorem: Let P be a SISO plant and K be a stabilizing controller. Then

$$\begin{aligned} \text{gain margin} &\geq \frac{1+b_{P,K}}{1-b_{P,K}}, \\ \text{phase margin} &\geq 2\arcsin(b_{P,K}). \end{aligned}$$

Proof: For SISO system at every ω

$$\begin{aligned} b_{P,K} &= \frac{1}{\|\ldots\|_\infty} \leq \frac{1}{\|\ldots\|} = \frac{|1+P(j\omega)K(j\omega)|}{\left\| \begin{pmatrix} 1 \\ K \end{pmatrix} \begin{pmatrix} 1 & P \end{pmatrix} \right\|} = \\ &= \frac{|1+P(j\omega)K(j\omega)|}{\left\| \begin{pmatrix} 1 \\ K \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 & P \end{pmatrix} \right\|} = \\ &= \frac{|1+P(j\omega)K(j\omega)|}{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K(j\omega)|^2}}. \end{aligned}$$

So at frequencies where $k := -PK \in R^+$ we have

$$\begin{aligned} b_{P,K} &\leq \frac{|1-k|}{\sqrt{(1+|P|^2)(1+k^2/|P|^2)}} \leq \\ &\leq \frac{|1-k|}{\sqrt{\min_P\{(1+|P|^2)(1+k^2/|P|^2)\}}} = \frac{|1-k|}{|1+k|} \end{aligned}$$

from which the gain margin result follows.

Similarly at frequencies where $PK = -e^\theta$

$$\begin{aligned} b_{P,K} &\leq \frac{|1-e^\theta|}{\sqrt{(1+|P|^2)(1+1/|P|^2)}} \leq \\ &\leq \frac{|1-e^\theta|}{\sqrt{\min_P\{(1+|P|^2)(1+1/|P|^2)\}}} = \\ &= \frac{2|\sin(\theta/2)|}{2} \end{aligned}$$

which implies the phase margin result.

Loop-Shaping Design

Recall from Lecture 2 that a good performance controller design requires

- in low frequency region:

  $\underline{\sigma}(PK) >> 1, \quad \underline{\sigma}(KP) >> 1, \quad \underline{\sigma}(K) >> 1.$

- in high frequency region:

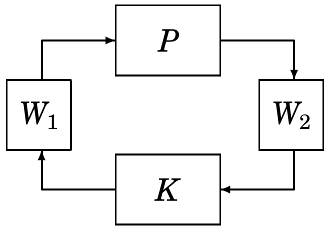
  $\overline{\sigma}(PK) << 1, \quad \overline{\sigma}(KP) << 1, \quad \overline{\sigma}(K) \leq M$

  where M is not too large.

Conclusion: Good performance depends strongly on the open loop shape.

H_∞ loop shaping design procedure was suggested by Glover and McFarlane, 1990. The idea is to use pre- and postcompensators which give a desired open loop shape.

Loop Shaping Procedure



1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.
2) Calculate $b_{opt}(P_s) = \sqrt{1 - \|\tilde{N}_s \;\; \tilde{M}_s\|_H^2}$. If it is small then return to Step 1 and adjust weights.
3) Select $\epsilon < b_{opt}(P_s)$ and design the H_∞ controller K_∞ such that

$$\left\| \begin{pmatrix} I \\ K_\infty \end{pmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty < \epsilon^{-1}.$$

4) The final controller is $K = W_1 K_\infty W_2$.

Remarks:

- In contrast to the classical loop shaping design we do not treat explicitly closed loop stability, phase and gain margins. Thus the procedure is simple.
- Observe that

$$\left\| \begin{pmatrix} I \\ K_\infty \end{pmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty =$$
$$= \left\| \begin{pmatrix} W_2 \\ W_1^{-1} K \end{pmatrix} (I + PK)^{-1} \begin{pmatrix} W_2^{-1} & PW_1 \end{pmatrix} \right\|_\infty .$$

  So it has an interpretation of the standard H_∞ optimization problem with weights.
- BUT!!! The open loop under investigation on Step 2 is $W_2 P W_1 K_\infty$ and $K_\infty W_2 P W_1$ whereas the actual open loop is given by $W_1 K_\infty W_2 P$ and $P W_1 K_\infty W_2$. This is not really what we has shaped!

Thus the method needs validation.

Justification of H_∞ Loop Shaping

We show that the degradation in the loop shape caused by K_∞ is limited.

Consider low-frequency region first.

$$\begin{aligned} \underline{\sigma}(PK) &= \underline{\sigma}(W_2^{-1} P_s K_\infty W_2) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_2)}, \\ \underline{\sigma}(KP) &= \underline{\sigma}(W_1 K_\infty P_s W_1^{-1}) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_1)} \end{aligned}$$

where κ denotes conditional number.

Thus small $\underline{\sigma}(K_\infty)$ might cause problem even if P_s is large. Can this happen?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\underline{\sigma}(K_\infty) \geq \frac{\underline{\sigma}(P_s) - \sqrt{\gamma^2 - 1}}{\sqrt{\gamma^2 - 1}\,\underline{\sigma}(P_s) + 1} \quad \text{if } \underline{\sigma}(P_s) > \sqrt{\gamma^2 - 1}.$$

Corollary: If $\underline{\sigma}(P_s) >> \sqrt{\gamma^2 - 1}$ then

$$\underline{\sigma}(K_\infty) \geq \frac{1}{\sqrt{\gamma^2 - 1}}.$$

Consider now high frequency region.

$$\begin{aligned}\overline{\sigma}(PK) &= \overline{\sigma}(W_2^{-1}P_sK_\infty W_2) \le \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_2),\\ \overline{\sigma}(KP) &= \overline{\sigma}(W_1K_\infty P_sW_1^{-1}) \le \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_1).\end{aligned}$$

Can $\overline{\sigma}(K_\infty)$ be large if $\overline{\sigma}(P_s)$ is small?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \ge 1/\gamma$ also satisfies

$$\overline{\sigma}(K_\infty) \le \frac{\sqrt{\gamma^2-1}+\overline{\sigma}(P_s)}{1-\sqrt{\gamma^2-1}\,\overline{\sigma}(P_s)} \quad \text{if } \overline{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2-1}}$$

Corollary: If $\overline{\sigma}(P_s) << 1/\sqrt{\gamma^2-1}$ then

$$\overline{\sigma}(K_\infty) \le \sqrt{\gamma^2-1}.$$

One can get the idea of proof from SISO relation

$$b_{P,K} \le \frac{|1+P_s(j\omega)K_\infty(j\omega)|}{\sqrt{1+|P_s(j\omega)|^2}\sqrt{1+|K_\infty(j\omega)|^2}}.$$

Denote

$$\overline{\sigma}_i = \overline{\sigma}(W_i), \quad \underline{\sigma}_i = \underline{\sigma}(W_i), \quad \kappa_i = \kappa(W_i).$$

Theorem: Let P be the nominal plant and let $K = W_1K_\infty W_2$ be the controller designed by loop shaping. Then if $b_{P_s,K_\infty} \ge 1/\gamma$ then

$$\begin{aligned}\overline{\sigma}(K(I+PK)^{-1}) &\le \gamma\overline{\sigma}(\tilde{M}_s)\overline{\sigma}_1\overline{\sigma}_2,\\ \overline{\sigma}((I+PK)^{-1}) &\le \min\{\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, 1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\},\\ \overline{\sigma}(K(I+PK)^{-1}P) &\le \min\{\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, 1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\},\\ \overline{\sigma}((I+PK)^{-1}P) &\le \frac{\gamma\overline{\sigma}(\tilde{N}_s)}{\underline{\sigma}_1\underline{\sigma}_2},\\ \overline{\sigma}((I+KP)^{-1}) &\le \min\{1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, \gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\},\\ \overline{\sigma}(P(I+KP)^{-1}K) &\le \min\{1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, \gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}\end{aligned}$$

where

$$\begin{aligned}\overline{\sigma}(\tilde{N}_s) &= \overline{\sigma}(N_s) = \left(\frac{\overline{\sigma}^2(P_s)}{1+\overline{\sigma}^2(P_s)}\right)^{1/2},\\ \overline{\sigma}(\tilde{M}_s) &= \overline{\sigma}(M_s) = \left(\frac{1}{1+\overline{\sigma}^2(P_s)}\right)^{1/2}.\end{aligned}$$

What have we learned today?

- H_∞ optimization of normalized coprime factors. Optimal value can be calculated via Hankel norm of the factors.
- Left or right coprime factors - does not matter.
- Stability margin $b_{P,K}$. The larger the better. Relation to gain and phase margins.
- H_∞ loop shaping via pre- and postcompensations and optimization of $b_{P,K}$.
- Relations PK, $KP \leftrightarrow P_s$, W.

Next lecture

- Gap Metric and ν-Gap Metric.
- Extended Loop Shaping.