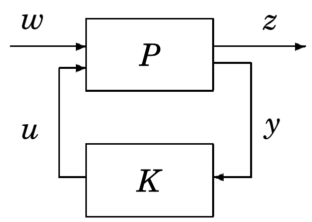
Lecture 6

- The H_∞ Optimization Problem
- Linear Quadratic Games
- Algebraic Riccati Equations
- State Space Solution to H_∞ Optimization

The H_∞ Optimization Problem



$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$T_{zw} = \mathcal{F}_l(P, K)$$

Optimal control:

$$\min_{K-\text{stab}} \|T_{zw}\|_\infty$$

Suboptimal control: Given γ find an internally stabilizing controller K such that

$$\|T_{zw}\|_\infty < \gamma.$$

The optimal control problem is solved by iterating γ in the suboptimal problem.

H_∞ Optimization in Frequency Domain

A good exposition can be found in the book [Francis, 1987].

The *Youla parameterization* of all internally stabilizing controllers gives an affine dependence of T_{zw} on the Youla parameter $Q \in RH_\infty$

$$T_{zw} = T_1 - T_2 Q T_3, \quad T_k \in RH_\infty$$

Thus the H_∞ optimization problem becomes

$$\min_{Q \in RH_\infty} \|T_1 - T_2 Q T_3\|_\infty$$

The optimization in Q is convex, but infinite-dimensional

H_∞ Optimization in Frequency Domain

In a special case, the H_∞ optimization problem is equivalent to

$$\min_{F \in RH_\infty} \|R - F\|_\infty = \text{dist}(R, RH_\infty)$$

where R is unstable.

This problem of approximating an L_∞ function by an H_∞ function is a classical problem from the beginning of the 20th century (Markov, Caratheodory, Fejer, Nevanlinna, Pick, Sarason and many others). Nehari solved it in 1957.

State Space Solution

Recall Linear Quadratic Control

If P satisfies the Riccati equation

$$A^TP + PA + Q - PBB^TP = 0$$

then every solution to $\dot{x} = Ax + Bu$ with $x(T) = 0$ satisfies

$$\begin{aligned}
\int_0^T [x^TQx + u^Tu]dt &= \int_0^T |u + B^TPx|^2 dt - 2\int_0^T (Ax+Bu)^TPxdt \\
&= \int_0^T |u + B^TPx|^2 dt - 2\int_0^T \dot{x}^TPxdt \\
&= \int_0^T |u + B^TPx|^2 dt - \int_0^T \frac{d}{dt}[x^TPx]dt \\
&= x(0)^TPx(0) + \int_0^T |u + B^TPx|^2 dt
\end{aligned}$$

with the minimizing control law $u = -B^TPx$.

A Linear Quadratic Game

If X satisfies the Algebraic Riccati Equation

$$A^TX + XA + Q - X(B_uB_u^T - B_wB_w^T/\gamma^2)X = 0$$

then $\dot{x} = Ax + B_uu + B_ww$ with $x(0) = x(T) = 0$ gives

$$\int_0^T [x^TQx + u^Tu - \gamma^2 w^Tw]dt = \int_0^T |u + B_u^TXx|^2 dt - \gamma^2 \int_0^T |w - B_w^TXx|^2 dt$$

This can be viewed as a dynamic game between the player u, who tries to minimize and w who tries to maximize.

The minimizing control law $u = -B_u^TXx$ gives

$$\int_0^T [x^TQx + u^Tu]dt \le \gamma^2 \int_0^T w^Twdt$$

so the gain from w to $z = (Q^{1/2}x, u)$ is at most γ.

Algebraic Riccati Equations

$$A^*X + XA + XRX + Q = 0$$

where $R = R^*$, $Q = Q^*$.

- The ARE is as important for control design as the Lyapunov equation is for system analysis.
- There are many solutions $X = X^*$ to ARE, the stabilizing one (which makes $A + RX$ stable) is unique!
- The ARE is a state space tool, which corresponds to factorization in frequency domain (recall spectral factorization in LQ Control).

How do we solve it?

Hamiltonian Matrix

Consider the $2n \times 2n$ matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^* \end{pmatrix}.$$

Lemma: Eigenvalues of H are symmetric with respect to the imaginary axis.

Proof: Introduce $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then $J^{-1}HJ = -H^*$, so λ is an eigenvalue of H if and only if $-\bar{\lambda}$ is.

In particular, if there are no purely imaginary eigenvalues then there are precisely n stable and n unstable eigenvalues of H.

Stable Invariant Subspace

Under assumption of no purely imaginary eigenvalues, let

$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in R^{2n\times n}$$

be a basis of the stable n-dimensional invariant subspace. Equivalently $HT = T\Lambda$ for some stable matrix $\Lambda \in R^{n\times n}$.

Lemma: If $\det(X_1) \neq 0$ then $X = X_2X_1^{-1}$ is a stabilizing solution to the ARE $A^*X + XA + XRX + Q = 0$

Proof: We are to prove

1) $X = X^*$.

2) X satisfies the ARE.

3) $A + RX$ is stable.

1) $HT = T\Lambda \Rightarrow T^*JHT = T^*JT\Lambda$. The matrix JH is symmetric then

$$T^*JT\Lambda = \Lambda^*T^*JT \quad \Leftrightarrow \quad T^*JT\Lambda - \Lambda^*T^*JT = 0.$$

So T^*JT satisfies the Lyapunov equation and Λ and $-\Lambda^*$ have no common eigenvalues. Hence $T^*JT = 0$ that is

$$X_2^*X_1 - X_1^*X_2 = 0 \quad \Leftrightarrow \quad X^* - X = 0.$$

2) & 3) Simple calculation gives

$$\begin{aligned} AX_1 + RX_2 &= X_1\Lambda, \\ -QX_1 - A^*X_2 &= X_2\Lambda. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} A + RX &= X_1\Lambda X_1^{-1} \\ -Q - A^*X &= X_2\Lambda X_1^{-1}. \end{aligned}$$

Thus $A + RX$ is stable and

$$XA + XRX = X_2\Lambda X_1 = -Q - A^*X$$

which implies the ARE.

How to solve the ARE

Under conditions

(H1) There are no pure imaginary eigenvalues of H.

(H2) $\det(X_1) \neq 0$ for some basis of stable invariant subspace.

we can find a stabilizing solution to ARE as follows:

1. Find a basis T for the stable invariant subspace, for example by Jordan decomposition. If (H1) holds, then it has the dimension n.
2. Partition T as
$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$
(H2) holds for some basis iff it holds for all basis.
3. Build $X = X_2X_1^{-1}$.

Notation

$H \in \mathrm{dom}(\mathrm{Ric})$ if (H1) and (H2) hold for H.

$X = \mathrm{Ric}(H)$ is the stabilizing solution to ARE.

ARE for H_∞ norm conditions

Let

$$G(s) = C(sI - A)^{-1}B$$

where (A, B, C) is stabilizable and detectable. Introduce the Hamiltonian matrix

$$H_0 = \begin{pmatrix} A & BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Theorem: Let $G \in RH_\infty$. The following conditions are equivalent:

1. $\|G\|_\infty < 1$,
2. (H1) holds for H_0,
3. $H_0 \in \mathrm{dom}(\mathrm{Ric})$.

Proof: For (2) $\Leftrightarrow$ (3), see [Zhou, p. 237]

(1) $\Leftrightarrow$ (2) The following conditions are equivalent:

1) $\|G\|_\infty < 1$

2) $\dfrac{\|G(j\omega)x\|^2}{\|x\|^2} < 1 \quad \forall x \neq 0,\ \omega$

3) $x^*[I - G(j\omega)^*G(j\omega)]x > 0, \quad \forall x \neq 0,\ \omega$

4) $I - G(j\omega)^*G(j\omega) > 0, \quad \forall \omega$

5) $\det(I - G(j\omega)^*G(j\omega)) \neq 0, \quad \forall \omega$

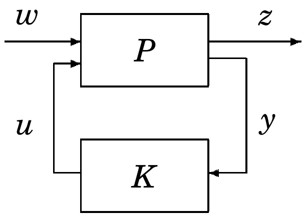
6) $\det(j\omega I - H_0) = \det \begin{pmatrix} j\omega I - A & -BB^* \\ C^*C & j\omega I + A^* \end{pmatrix} =$

$= \det(j\omega I - A)\det(j\omega I + A^*)\det(I - G^*G) \neq 0$

$\forall \omega.$

Thus we can describe the condition $\|G\|_\infty < 1$ by existence of the stabilizing solution to an ARE.

Assumptions



$$P = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A1) (A, B_w, C_z) is stabilizable and detectable,

(A2) (A, B_u, C_y) is stabilizable and detectable,

(A3) $D_{zu}^* \begin{pmatrix} C_z & D_{zu} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$,

(A4) $\begin{pmatrix} B_w \\ D_{yw} \end{pmatrix} D_{yw}^* = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

State Space H_∞ optimization

The solution involves two AREs with Hamiltonian matrices

$$
\begin{aligned}
H_\infty &= \begin{pmatrix} A & \gamma^{-2}B_wB_w^* - B_uB_u^* \\ -C_z^*C_z & -A^* \end{pmatrix} \\
J_\infty &= \begin{pmatrix} A^* & \gamma^{-2}C_z^*C_z - C_y^*C_y \\ -B_wB_w^* & -A \end{pmatrix}
\end{aligned}
$$

Theorem: There exists a stabilizing controller K such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$,
2. $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$,
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, one such controller is

$$
K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]
$$

where

$$
\hat{A}_\infty = A + \gamma^{-2}B_wB_w^*X_\infty + B_uF_\infty + Z_\infty L_\infty C_y,
$$

$$
F_\infty = -B_u^*X_\infty, \;\; L_\infty = -Y_\infty C_y^*,
$$

$$
Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.
$$

Furthermore, the set of all stabilizing controllers such that $\|T_{wz}\|_\infty < \gamma$ can be explicitly obtained as lower LFT (see [Zhou,p. 271]).

[Doyle J., Glover K., Khargonekar P., Francis B., *State Space Solution to Standard H^2 and H^∞ Control Problems*, IEEE Trans. on AC **34** (1989) 831–847.]

Idea of Proof

The dynamic game viewpoint gives a solution in the case of full information, where both state and disturbance are measurable. This gives the first ARE.

This can be combined with a “worst case observer”, finding the smallest disturbance compatible with available measurements. This gives the second ARE.

Combining the full information solution with the worst case observer, solves the dynamc game problem with limited measurement information, provided that the spectral radius condition holds.

What have we learned today?

- H_∞ optimization is fundamental problem for robust synthesis.
- A dynamic game between controller and disturbance
- The state space approach gives easily implementable conditions and formulas.
- Algebraic Riccati Equation is the main computational tool.