Lecture 3

- Examples: bicycle, pendulum, four tank process
- Complex analysis
- Bode's relations
- Bode's integral
- Sensitivity bounds
- Examples revisited

Examples

Why are some bicycles impossible to ride?

How short inverted pendulums can be balanced by hand?

What is the mechanism behind the unstable zero in the four tank process?

Unstable poles

An unstable pole means that the response without input grows exponentially as e^{pt}. It is intuitively clear that in order to stabilize the system, the feedback loop must be faster than the time constant $1/p$. This gives the cutoff frequency constraint

$$\omega_c \gtrsim p$$

A formal argument will be given later

Unstable system with time-delay

A time-delay T means that control action at time t does not have any effect until time $t+T$. Hence, it is intuitively clear that an unstable pole can not be stabilized unless

$$T \lesssim \frac{1}{p}$$

A formal argument will be given later

Unstable zeros

An unstable zero z sometimes results in a step response that initially goes in the wrong direction. The time constant of such dynamics is $1/z$ and limits the speed of control:

$$\omega_c \lesssim z$$

A formal argument will be given later

Mini-problem

Does any of the criteria above apply to the bicycle or to the pendulum?

Some Facts from Complex Analysis

1) *D'Alembert-Euler-Cauchy-Riemann condition*.

Let $z = x + jy$ and $f(z) = u(x,y) + jv(x,y)$. Then f is analytical at z iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

By this condition

- one can determine v by u and vice versa

$$v(x,y) = \int_{z_0}^{z} \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right) + C.$$

- assumption $f(z_0) \in R$ gives $C = 0$.
- u and v are harmonic functions, i.e. $\Delta u = \Delta v = 0$.

2) *The Poisson integral*.

For "any" harmonic in RHP function u and for all $x + jy$ in RHP

$$u(x+jy) = \frac{1}{\pi} \int_{-\infty}^{+\infty} u(j\omega) \frac{x\, d\omega}{x^2 + (y-\omega)^2}.$$

Proof: csd.newcastle.edu.au/appendices/appendixC_8_1.html
Corollary (Schwarz integral): For "any" f analytical in RHP

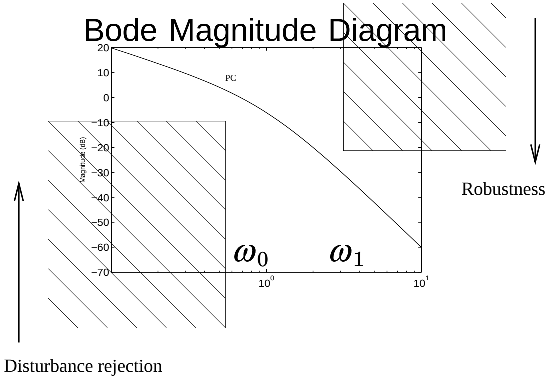
$$f(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \text{Re}\{f(j\omega)\} \frac{d\omega}{z - j\omega} + jC \qquad \text{for Re}\, z > 0$$

Furthermore, if f is analytical and has no zeros in RHP then $\ln f$ is also analytical and for some $|c_0| = 1$

$$f(z) = c_0 \exp\left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} \ln|f(j\omega)| \frac{d\omega}{z - j\omega} \right\}.$$

Small warning sign: Convergence issues might arise in the formulas on this page. If u, f, $\ln|f|$ are bounded on large semicircles in the RHPL, you should be fine.

A frequency domain specification



The shaded areas are “forbidden areas”. For how small interval $[\omega_0, \omega_1]$ can the specification be satisfied?

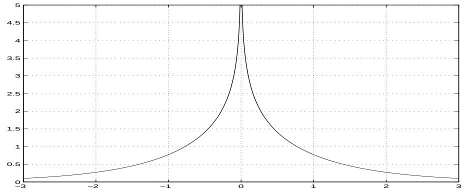
Bode’s Gain and Phase Relation

These constraints arise from the internal stability requirement. For simplicity we shall assume that both P and K are scalar.

Let L be analytical and minimum phase function in RHP and $L(0) > 0$. Then

$$\arg L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d \ln |L(j\omega_0 e^{\nu})|}{d\nu} \ln \coth \frac{|\nu|}{2} \, d\nu.$$

$\ln \coth |\nu|/2$



- $\frac{d \ln |L|}{d\nu}$ is the slope of Bode plot (generally negative).
- If L attenuates slowly (rapidly) near ω_0 then $\arg L(j\omega_0)$ is large (small). For example, if $d \ln |L|/d\nu = -c$ then

$$\arg L(l\omega_0) = -\frac{c}{\pi} \int_{-\infty}^{+\infty} \ln \coth \frac{|\nu|}{2} \, d\nu = -\frac{c\pi}{2}.$$

- If $|L(j\omega_c)| = 1$ then $\pi + \arg L(j\omega_c)$ is the phase margin and

$$|1 + L(j\omega_c)| = |1 + L(j\omega_c)^{-1}| = 2 \left| \sin \frac{\pi + \arg L(j\omega_c)}{2} \right|.$$

  must not be small. So it is important to keep the slope of L near ω_c not much smaller than -1.
- There is a generalization of Bode’s gain and phase relation to the case of nonminimum phase function L (see [Zhou,p. 96]).

Theorem: Bode’s Sensitivity Integral

Let $\{p_k\}_{k=1}^{K}$ denote the set of unstable poles of L. Assume that the relative degree of L is at least 2. Then

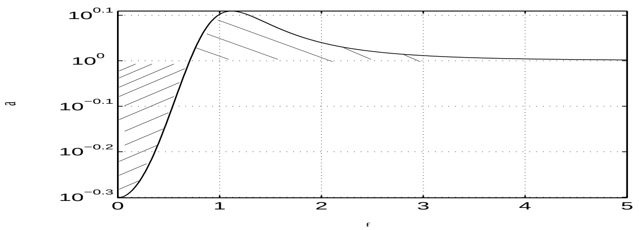
$$\int_0^{+\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{k=1}^{K} \mathrm{Re}\, p_k.$$

Proof for stable S:
By Poisson integral formula with $y = 0$ we have

$$\begin{aligned} &\int_0^{+\infty} \ln |S(j\omega)| \, d\omega \\ &= \int_0^{+\infty} \ln |S(j\omega)| \lim_{x\to\infty} \frac{x^2 \, d\omega}{x^2 + \omega^2} = \lim_{x\to\infty} \int_0^{+\infty} \ln |S(j\omega)| \frac{x^2 \, d\omega}{x^2 + \omega^2} \\ &= \lim_{x\to\infty} \frac{\pi}{2} x \ln |S(x)| = -\lim_{x\to\infty} \frac{\pi}{2} x \ln |1 + L(x)| = -\lim_{x\to\infty} \frac{\pi}{2} x L(x) = 0 \end{aligned}$$

Bode's integral formula

- Can the sensitivity be small for all frequencies?
  - No, we have $S(\infty) = 1$!
- The "water-bed effect". Push the curve down at one frequency and it pops up at another!



Another fact from complex analysis

3) *The Maximum Modulus Theorem*.
Suppose that the function f is analytic in a set containing the unit disc. Then

$$\max_{|z|\leq 1} |f(z)| \quad = \quad \max_{|z|=1} |f(z)|$$

Corollary.
Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\max_{\text{Re } s\geq 0} |G(s)| \quad = \quad \max_{\omega\in\mathbf{R}} |G(i\omega)|$$

Sensitivity bounds from unstable zeros and poles

The sensitivity must be one at an unstable zero:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := [1 - C(z)P(z)]^{-1} = 1$$

The complimentary sensitivity must be one at an unstable pole:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := C(p)P(p)[1 - C(p)P(p)]^{-1} = 1$$

Performance limitations from unstable zeros

Note that (for stable W_s and S)

$$\sup_{\omega} \|W_s(i\omega)S(i\omega)\| = \sup_{\text{Re } s\geq 0} \|W_s(s)S(s)\|$$

so the specification

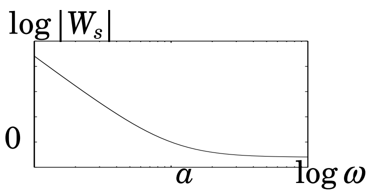
$$\|W_s(i\omega)S(i\omega)\| \leq 1 \qquad\qquad \text{for all } \omega$$

can not be met unless $\|W_s(z_i)\| \leq 1$ for unstable zeros z_i of P.

In particular, if

Let $W_s(s) = \dfrac{s+a}{2s}$

then z_i must be $\geq a$



Performance limitations from unstable poles

Note that (for stable W_t and S)

$$\sup_{\omega} \|W_t(i\omega)T(i\omega)\| = \sup_{\text{Re } s\geq 0} \|W_t(s)T(s)\|$$

so the specification

$$\|W_t(i\omega)T(i\omega)\| \leq 1 \qquad\qquad \text{for all } \omega$$

can not be met unless $\|W_p(p_i)\| \leq 1$ for unstable zeros p_i of P.

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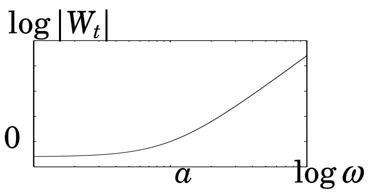


Unstable zero and unstable pole

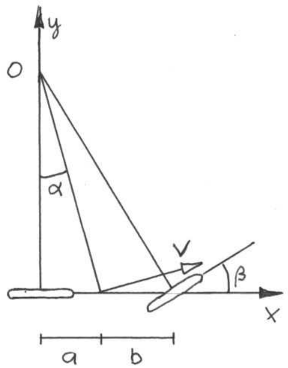
Let $P = (s-z)(s-p)^{-1}\widehat{P}$, with $\widehat{P}$ proper and $\widehat{P}(p) \neq 0$.

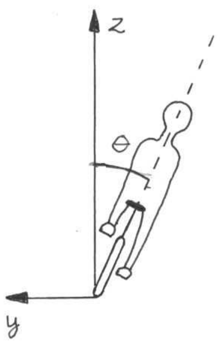
Then, for stable closed loop systems the sensitivity function satisfies

$$\begin{aligned}
\sup_{\omega} |S(i\omega)| &= \sup_{\omega} \left|\frac{1}{1+CP}\right| = \sup_{\omega} \left|\frac{1}{1+C(i\omega-z)(i\omega-p)^{-1}\widehat{P}}\right| \\
&= \sup_{\omega} \left|\frac{i\omega-p}{i\omega-p+C(i\omega-z)\widehat{P}}\right| = \sup_{\omega} \left|\frac{i\omega+p}{i\omega-p+C(i\omega-z)\widehat{P}}\right| \\
&= \sup_{\text{Re } s\geq 0} \left|\frac{s+p}{s-p+C(s-z)\widehat{P}}\right| \geq \left|\frac{z+p}{z-p}\right|
\end{aligned}$$

so the sensitivity function must have a high peak *for every controller if* $p \approx z$.

Bicycle Tilt Dynamics





$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

The Rear Wheel Steered Bike

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed:	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g =$ ms^{-2}

The transfer function from β to θ is $P(s) = \frac{mV_0\ell}{b}\frac{as+V_0}{Js^2-mg\ell}$

The system has unstable pole p and a zero z

$$p^{-1} = \sqrt{\frac{J}{mg\ell}} \approx 0.4 \text{ s} \qquad\qquad z^{-1} = -\frac{a}{V_0} \approx 0.06\text{s}$$

What have we learned today?

Complex analysis provides very a powerful tool to understand how the open loop plant dynamics limit the achievable closed loop performance

- Bode's relations
- Bode's integral
- Sensitivity bounds