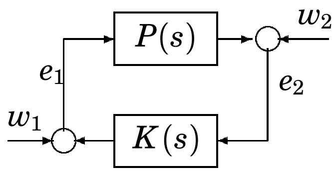
Lecture 2

- Well-posedness and internal stability.
- Coprime factorization over H_∞.
- Performance specifications in terms of H_2 and H_∞ norms.

Well-Posedness

Even for a matrix equation $Ax = b$, the solution x does not always exist.

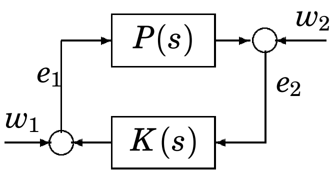
Feedback gives a linear equation in an infinite-dimensional space. Solvability?



$$e_1 = Ke_2 + w_1$$
$$e_2 = Pe_1 + w_2$$

Example: Let $P(s) = \frac{s+1}{s+2}$ and $K(s) = 1$. The closed-loop system is not proper

$$\frac{1}{1 - \frac{s+1}{s+2}} = \frac{s+2}{s+2-s-1} = s+2.$$



$$\begin{pmatrix} I & -K \\ -P & I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The system is solvable if the matrix of the system is invertible for almost all s. Then

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} I & -K \\ -P & I \end{pmatrix}^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Definition: The closed-loop system is called *well-posed* if

$$\begin{pmatrix} I & -K \\ -P & I \end{pmatrix}^{-1}$$

exists for almost all s and is a proper function.

Lemma: Let G be proper and square. Then G^{-1} exists for almost all s and is proper if and only if $G(\infty)$ is nonsingular.

Proof: Let $G(s) = C(sI - A)^{-1}B + D$. Hence $G(\infty) = D$.

"$\Rightarrow$"

G^{-1} exists and is proper $\Rightarrow$ $G(\infty)^{-1}$ exists and is bounded $\Rightarrow$ $G(\infty)$ is nonsingular.

"$\Leftarrow$"

Calculate the inverse by [Zhou,p. 14]

$$\begin{aligned} G(s)^{-1} &= (D + C(sI - A)^{-1}B)^{-1} = \\ &= D^{-1} - D^{-1}C(sI - A + BD^{-1}C)^{-1}BD^{-1}. \end{aligned}$$

Hence, the inverse exists for almost all s (except the eigenvalues of the matrix $A - BD^{-1}C$) and is proper.

Corollary: The following statement are equivalent

1. The closed-loop system (P, K) is well-posed,
2. $\begin{pmatrix} I & -K(\infty) \\ -P(\infty) & I \end{pmatrix}$ is invertible,
3. $I - K(\infty)P(\infty)$ is invertible,
4. $I - P(\infty)K(\infty)$ is invertible.

Proof: Due to [Zhou,p. 14] and $\det(I) = 1$ we have

$$\det \begin{pmatrix} I & -K \\ -P & I \end{pmatrix} = \det(I - KP) = \det(I - PK)$$

Remark: Very often in practical cases we have $P(\infty) = 0$ (no direct feed-through). This gives well-posedness automatically

Internal Stability

Well-posedness guarantees solvability. What about stability?

Definition: The closed-loop system is called *internally stable* if

$$\begin{pmatrix} I & -K \\ -P & I \end{pmatrix}^{-1} \in RH_\infty$$

The H_∞-norm of this operator is the L_2-gain from disturbances w to loop signals e. Using the formula in [Zhou,p. 14] we get the equivalent condition

$$\begin{pmatrix} (I - KP)^{-1} & K(I - PK)^{-1} \\ P(I - KP)^{-1} & (I - PK)^{-1} \end{pmatrix} \in RH_\infty.$$

Corollary 1: Let $K \in RH_\infty$. Then (P, K) is internally stable iff it is well-posed and $P(I - KP)^{-1} \in RH_\infty$.

Corollary 2: Let $P \in RH_\infty$. Then (P, K) is internally stable iff it is well-posed and $K(I - PK)^{-1} \in RH_\infty$.

Corollary 3: Let P and $K \in RH_\infty$. Then (P, K) is internally stable iff it is well-posed and $(I - PK)^{-1} \in RH_\infty$.

See [Zhou,p.69] for proof (very easy).

Theorem

The system is internally stable if and only if it is well-posed and

1. There is no unstable pole-zero cancellation in PK,
2. $(I - PK)^{-1} \in RH_\infty$.

Proof sketch: If there is an unstable pole-zero cancellation in PK, then this means that there is an unstable mode of K which does not show up in the output of P and hence can not be stabilized by feedback from P.

Stability of $PK(I - PK)^{-1} = -I + (I - PK)^{-1} = (I - PK)^{-1}PK$ and unstable pole-zero cancellation in the product PK directly implies stability of $K(I - PK)^{-1}$ and $(I - PK)^{-1}P$.

To prove stability of $(I - KP)^{-1}$ is harder.

Definition: Let $m,\ n\ \in\ RH_\infty$. Then m and n are said to be *coprime over* RH_∞ if there exist $x,\ y\ \in\ RH_\infty$ such that $xm + yn = 1$.

Definition: Two matrices $M,\ N \in RH_\infty$ are said to be

- *right coprime over* RH_∞ if there exist $X,\ Y \in RH_\infty$ such that
$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = XM + YN = I.$$
- *left coprime over* RH_∞ if there exist $X,\ Y \in RH_\infty$ such that
$$\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = MX + NY = I.$$

The right hand equations are called *Bezout identities*

Coprime Factorization over RH_∞

Let P be a proper real rational matrix. A right coprime factorization (rcf) of P is a factorization $P = NM^{-1}$ where N and M are right coprime over RH_∞.

Similarly, a left coprime factorization (lcf) of P has the form $P\ =\ \tilde{M}^{-1}\tilde{N}$ and $\tilde{N}$ and $\tilde{M}$ are left coprime over RH_∞. Of course, M and $\tilde{M}$ are square.

- Coprimeness means there is no cancellation in the fraction (no nontrivial common right/left divisors).
- For scalar plant rcf=lcf.
- For real rational matrices both factorizations always exist.
- They are not unique.
- There is a state space method to calculate them.

Feedback Interpretation

Let $P(s) = C(sI - A)^{-1}B + D$, that is

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned}$$

Introduce a change of control $v\ =\ u - Fx$ where $A + BF$ is stable. Then we get

$$\begin{aligned} \dot{x} &= (A + BF)x + Bv \\ y &= (C + DF)x + Dv \end{aligned} \qquad\qquad u = Fx + v$$

Denote by $M(s)$ the transfer function from v to u and by $N(s)$ the transfer function from v to y

$$\begin{aligned} M(s) &= F(sI - A - BF)^{-1}B + I, \\ N(s) &= (C + DF)(sI - A - BF)^{-1}B + D. \end{aligned}$$

Therefore, $u = Mv$, $y = Nv$ and, finally, $y = NM^{-1}u$

Coprime Factorization and Internal Stability

Consider a plant P and a controller K with some rcf and lcf

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N} \qquad\qquad K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$$

Theorem: The following conditions are equivalent:

1. The closed-loop system (P, K) is internally stable.
2. $\begin{pmatrix} M & U \\ N & V \end{pmatrix}$ is invertible in RH_∞.
3. $\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ is invertible in RH_∞.
4. $\tilde{M}V - \tilde{N}U$ is invertible in RH_∞.
5. $\tilde{V}M - \tilde{U}N$ is invertible in RH_∞.

Proof: See [Zhou,p. 74].

Double Coprime Factorization

A double coprime factorization (dcf) of P over RH_∞ is a factorization

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

such that there exist X_r, X_l, Y_r, $Y_l \in RH_\infty$ and it holds

$$\begin{pmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & -Y_l \\ N & X_l \end{pmatrix} = I.$$

- The only difference between the dcf and a couple of some rcf and lcf is in additional condition $X_r Y_l = Y_r X_l$
- The controller $K = -Y_l X_l^{-1} = -X_r^{-1} Y_r$ is internally stabilizing.
- There is a state space method to calculate dcf explicitly (see [Zhou]).

Performance Specifications

Introduce the following notations

$$\begin{aligned} L_i &= KP, & L_o &= PK, \\ S_i &= (I + L_i)^{-1}, & S_o &= (I + L_o)^{-1}, \\ T_i &= I - S_i, & T_o &= I - S_o. \end{aligned}$$

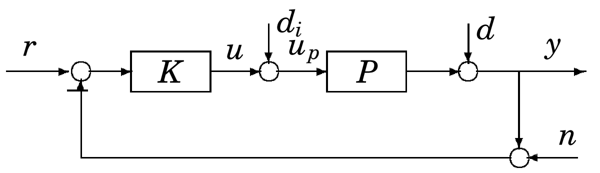
L_i — the input loop transfer function,

L_o — the output loop transfer function,

S_i — the input sensitivity ($u_p = S_i d_i$).

S_o — the output sensitivity ($y = S_o d$).

T — the complementary sensitivity.



$$\begin{aligned} y &= T_o(r-n) + S_o P d_i + S_o d, \\ r - y &= S_o(r-d) + T_o n - S_o P d_i, \\ u &= KS_o(r-n) - KS_o d - T_i d_i, \\ u_p &= KS_o(r-n) - KS_o d + S_i d_i \end{aligned}$$

1) Good performance requires

$$\underline{\sigma}(L_o) >> 1, \quad \underline{\sigma}(L_i) >> 1, \quad \underline{\sigma}(K) >> 1.$$

2) Good robustness and good sensor noise rejection requires

$$\overline{\sigma}(L_o) << 1, \quad \overline{\sigma}(L_i) << 1, \quad \overline{\sigma}(K) \leq M.$$

Conflict!!! Separate frequency bands!

H_2 and H_∞ Performance.

For good rejection of d at y and u both $\|S_o\|$ and $\|KS_o\|$ should be small at low-frequency range. It can be captured by the norm specification

$$\left\| \begin{pmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{pmatrix} \right\|_{2 \text{ or } \infty} \leq 1$$

where W_d reflects the frequency contents of d or models the disturbance power spectrum, W_e reflects the requirement on the shape of S_o and W_u reflects restriction on the control.

For robustness to high frequency uncertainty, the complimentary sensitivity has to be limited

$$\left\| \begin{pmatrix} W_e S_o W_d \\ \rho W_u T_o W_d \end{pmatrix} \right\|_\infty \leq 1$$

What have we learned today?

- Well-posedness to guarantee solvability.
- Internal stability — stability of a feedback loop
- Coprime factorization and internal stability.
- State space formula to calculate coprime factors.
- Performance specifications
- Using norms to capture loop requirements.