Robust Control, 6p

- Introduction. Spaces, operators, norms.
- Internal stability, performance measures
- Fundamental limitations
- Models of system uncertainty
- Structured uncertainty and μ-synthesis
- H_2 and H_∞ optimal control
- Gap metrics and H_∞ loop shaping

Lecture 1

- Historical remarks
- The class of linear systems as a linear space
- Norm and inner product as a way to measure distance
- Banach and Hilbert spaces: L_∞. and L_2
- The Hardy spaces: H_2 and H_∞
- Matrix computations

Introduction

- Without uncertainty there is no need for feedback
- A brief history
  - Black, Bode and Nyquist
  - Bode's ideal loop transfer function
  - Horowitz and QFT
  - State space theory
  - $\mathcal{H}_\infty$, Zames, Glover, Doyle, ...
- How to cope with uncertainty
  - Live with it: Robust control!
  - Reduce it: Adaptive control!

The Feedback Amplifier

The repeater problem

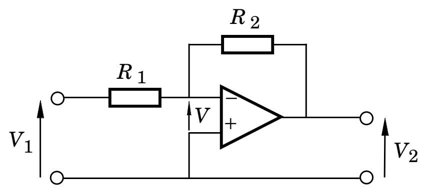
Blacks invention 1927

Nyquist 1932

Blacks paper 1934

Bode 1940

Bodes book 1945



$$\frac{V_2}{V_1} = -\frac{R_2}{R_1}\frac{1}{1+\frac{1}{A}(1+\frac{R_2}{R_1})}$$

Early theoretical insights

- Nyquist 1932
- Bode 1940
- Important ideas

    Nyquist curve
    Bode diagram
    Bodes relations
    Bodes integrals
    Bodes ideal loop transfer function

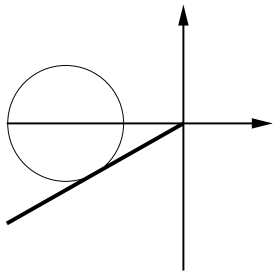
- Horowitz 1963 +

    Templates
    Qualitative Feedback Theory (QFT)

Bodes Ideal Loop Transfer Function

The repeater problem. Large gain variations in tube amplifiers What should a transfer function look like to be independent of gain?

$$L(s) = \left(\frac{s}{\omega_{gc}}\right)^n$$



Phase margin invariant with loop gain
$n = -1.5$ gives $\varphi_m = 45^\circ$

Horowitz extended Bodes ideas to deal with arbitrary plant variations not just gain variations in the QFT method.

State Space Theory

- Many useful concepts

    State
    Observability, reachability
    Kalman filters and separation

- Uncertainty as parameter errors or additive disturbances
- Difficult to deal with unmodeled dynamics
- Multi-variable systems

    Singular values are what matters for robustness!

- H_∞ theory

    Brought uncertainty into the picture again!
    Structured uncertainty and μ

What is this course about?

We design a controller C for a mathematical model M and want the corresponding real process P to behave well.

Problems:

- $P \neq M$
- Even if $P = M$ there is controller implementation errors

Robustness philosophy: The controller C is *robust* if

$$\begin{array}{ccc} P & \approx & M \\ C_r & \approx & C \end{array} \quad \Rightarrow \quad (P, C_r) \approx (M, C).$$

- What does it mean "$\approx$"? (This lecture)
- How to check this? — Analysis.
- How to find the controller? — Synthesis

Linear (or vector) space

Dream: To use intuition from $\mathbb{R}^n$ in more general situations

Consider a set $X = \{x\}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with two operations $+ : X \times X \to X$ and $\cdot : \mathbb{F} \times X \to X$. Then X is a linear space if

1. $x_1 + x_2 = x_2 + x_1$.
2. $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$.
3. $\exists 0 \in X$ such that $x + 0 = x \ \forall x \in X$.
4. $\forall x \in X \ \exists(-x) \in X$ such that $x + (-x) = 0$.
5. $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$.
6. $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2$.
7. $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$.
8. $1x = x$.

The space of linear systems

Denote by $\mathcal{L}$ the set of all linear systems. It becomes the linear space with the following natural definition of $+$ and $\cdot$

$$\begin{array}{rcl} y_1 & = & G_1 u, \\ y_2 & = & G_2 u \end{array} \Rightarrow (G_1 + G_2)u = y_1 + y_2,$$

$$y = Gu \ \Rightarrow \ (\lambda G)u = \lambda y.$$

Only algebraic linearity is rather poor generalization of $\mathbb{R}^n$. What about the distance between two linear systems? What does it mean

$$G_1 \approx G_2?$$

Normed linear space

A linear space X is called *normed* if every vector $x \in X$ has an associated real number $\|x\|$ — its "length", called the norm of the vector x, — with the following properties

1. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$.
3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.

Now we can say that $G_1 \approx G_2$ if $\|G_2 - G_1\|$ is small.

How should one equip the space $\mathcal{L}$ with a norm? A good chice should support understanding, but also allow for computational analysis and synthesis.

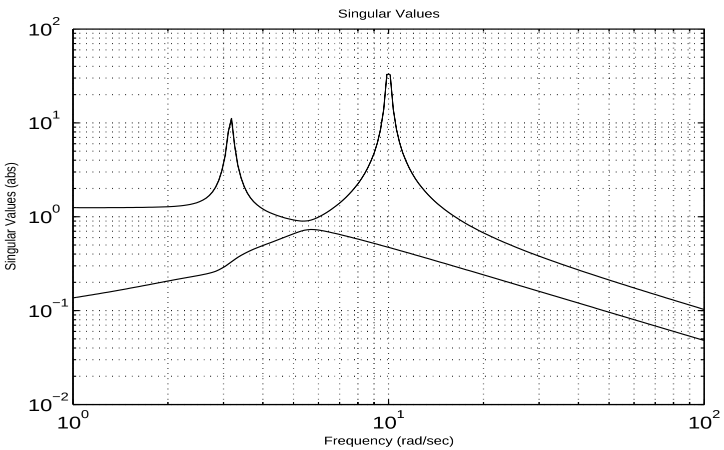
Induced norm

A linear system can be considered as an operator from input U to output Y. If U and Y are normed linear spaces then the following system norm is said to be *induced* by the signal norms on U and Y

$$\|G\| = \sup_{\|u\|_U \leq 1} \|Gu\|_Y.$$

Geometrical sense: $\|G\|$ is the maximal possible gain of the unit input.

Singular value plot for 2×2 system



What does the plot tell you?

Banach and Hilbert spaces

The space $\mathbb{R}^n$ also has an *inner product*, that is a functional $\langle , \rangle$ with the properties

1. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.
2. $\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$.
3. $\langle x_1 + x_2, x_3 \rangle = \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle$.
4. $\langle \lambda x_1, x_2 \rangle = \lambda \langle x_1, x_2 \rangle$.

If there is an inner product on X then the norm can be defined as

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{1}$$

A *complete* normed linear space is called Banach space. A Banach space with inner product and the norm (1) is called Hilbert space.

Remarks:

- Completeness means that there is no holes in the space. It is very important property. For example, people deal with real numbers rather than with rational numbers because the latter is not the complete space.
- Existence of the inner product gives an additional nice property of the corresponding norm which makes the space be very similar to $\mathbb{R}^n$. This property is

  $$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2(\|x_1\|^2 + \|x_2\|^2).$$

  It simplifies drastically the optimization in Hilbert spaces.

Examples: L_2 and L_∞ spaces.

Example 1: L_2 space. Consider the linear space of all matrix-valued functions on $\mathbb{R}$

$$L_2(\mathbb{R}) = \{F \ : \ \int_{\mathbb{R}} \mathrm{tr}[F(t)^* F(t)]\, dt < +\infty\}.$$

This is the Hilbert space with the inner product

$$\langle f, g \rangle_2 = \int_{\mathbb{R}} \mathrm{tr}[f(t)^* g(t)]\, dt$$

Example 2: L_∞ space. Consider the linear space of all matrix-valued functions on $\mathbb{R}$

$$L_\infty(\mathbb{R}) = \{F \ : \ \text{ess sup}\, \sigma_{max}[F(t)] < +\infty\}.$$

This is a Banach space with $\|F\|_\infty = \text{ess sup}_{t \in \mathbb{R}}\, \sigma_{max}[F(t)]$

Choice of U and Y as L_2.

One of the simplest choices of the input and output spaces is $L_2(\mathbb{R})$ mainly because it is the Hilbert space. In this case the linear system G is a linear operator on L_2

$$G\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$$

and the norm of the linear system is L_2-induced norm

$$\|G\| = \sup_{\|u\|_2 \le 1} \|Gu\|_2 = \|G(j\omega)\|_\infty$$

where $G(s)$ is the transfer function of LTI system (Parseval's relation + Theorem 4.3 in [Zhou+Doyle]).

Remark: Choosing the spectral norm in U and the power norm in Y one can get

$$\|G\| = \|G(j\omega)\|_2$$

(see [Zhou+Doyle+Glover] for details).

Stability and Hardy spaces.

Stability is a very important issue in system analysis. Therefore, it is also important to capture stability in the relation $\approx$. In other words, if G_1 is stable and G_2 is not then $\|G_1 - G_2\|$ must be large.

This motivates the introduction of *Hardy spaces*:

Define for $p = 2$ and $p = \infty$

$$\begin{aligned} H_p &= \{F \in L_p(j\mathbb{R}) :\ F \text{ is analytic in the right half plane}\} \\ \|F\|_{H_p} &= \sup_{\sigma > 0} \|F(\sigma + j\omega)\|_{L_p}. \end{aligned}$$

Thus if G_1 is stable and $\|G_1 - G_2\|_{H_p}$ is finite then G_2 is also stable.

Are these norms easy to compute?

If G is stable then

$$\|G\|_p := \|G(j\omega)\|_{L_p} = \|G\|_{H_p}.$$

L_2/H_2 norm:

For rational G, the norm $\|G\|_2$ can be finite only if G is strictly proper

Theorem 1: Let $G(s) = C(sI - A)^{-1}B$ and A is stable matrix. Then

$$\|G\|_2^2 = \operatorname{tr}(B^*QB) = \operatorname{tr}(CPC^*)$$

where P is controllability and Q is observability Gramian

$$\begin{aligned} AP + PA^* + BB^* &= 0, \\ A^*Q + QA + C^*C &= 0. \end{aligned}$$

The formula for $\|G\|_2$

The transfer function $G(s)$ is the Laplace transform of the impulse response

$$g(t) = \begin{cases} Ce^{At}B, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Hence by Parseval's formula

$$\begin{aligned} \|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\{G(i\omega)^* G(i\omega)\} d\omega = \int_0^\infty \text{trace}\{g(t)^* g(t)\} dt \\ &= \int_0^\infty \text{trace}\{B^* e^{A^*t} C^* C e^{At} B\} dt = \operatorname{tr}(B^*QB) \end{aligned}$$

since

$$Q = \int_0^\infty e^{A^*t} C^* C e^{At} dt$$

L_∞/H_∞ **norm:**

For real-rational plants $\|G\|_\infty < +\infty$ only if $G(s)$ is proper.

The computation is more complicated than for H_2 norm and requires a search.

Theorem 2: Let $G(s) = C(sI - A)^{-1}B + D \in L_\infty$. Then $\|G\|_\infty < \gamma$ if and only if

1. $\sigma_{max}(D) < \gamma$,
2. H has no eigenvalues on the imaginary axis

where $R = \gamma^2 I - D^*D$ and

$$H = \begin{pmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{pmatrix}$$

$\|G\|_\infty$ when $G(s) = C(sI - A)^{-1}B + D$

Let γ^2 be an eigenvalue of $G(i\omega)G(i\omega)^*$ with eigenvector u:

$$[C(i\omega I - A)^{-1}B + D]^*u = \gamma v \qquad [C(i\omega I - A)^{-1}B + D]v = \gamma u$$

Define

$$p = (i\omega I - A)^{-1}Bu \qquad q = (-i\omega I - A^*)^{-1}C^*v$$

Then

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^* \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$i\omega \begin{bmatrix} p \\ q \end{bmatrix} = \underbrace{\left\{ \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} + \begin{bmatrix} B & 0 \\ -C^* & 0 \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^* \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} \right\}}_{H} \begin{bmatrix} p \\ q \end{bmatrix}$$

Hence H must have a purely imaginary eigenvalue.

What have we learned today?

- Robustness as a property of the closed-loop system to have similar behavior for all plants “close” to the nominal one.
- Normed linear space as the main tool to handle “close-far” notion. G_1 is “close” to $G_2 \leftrightarrow \|G_1 - G_2\|$ is small.
- $\|G\|$ depends on norms of input and output signal spaces.
- L_2 and L_∞ plus stability gives H_2 and H_∞. This is the most important choices for $\|G\|$.
- They are also not very hard to compute — H_2 easier, H_∞ harder (needs an iteration).