# The Least Measure of a Matrix Set with Applications

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# Outline

- Background and Motivation
- The Least Measure
- Performance Estimate
- Summary

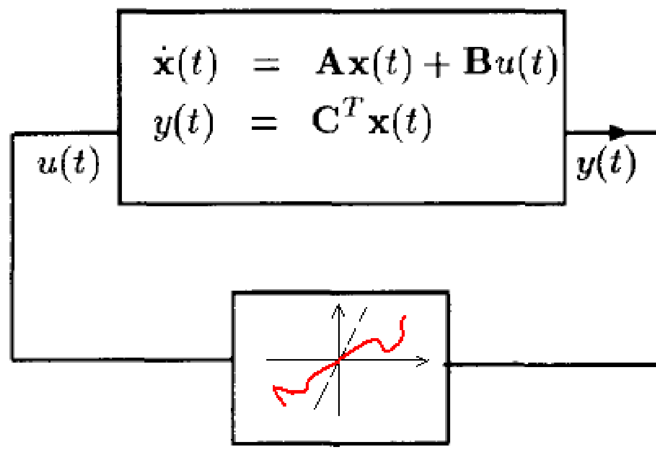
# Absolute Stability for Lur'e Systems

- A Lur′e system is a linear plant with a sector-bounded nonlinear output feedback
- $[k_1,k_2]$-absolute stability means global asymptotic stability with respect to the sector $[k_1,k_2]$
- Problem: Determine the largest sector bound that guarantees absolute stability
- Sub-problem: Given the sector bound, verify whether the system is absolutely stable or not

$$\dot{x}(t) = Ax(t) + b\varphi(y,t)$$

$$y(t) = cx(t)$$

$$k_1 y^2 \leq y\varphi(y,t) \leq k_2 y^2$$



# Classical Criteria

Frequency Domain

- Circle Criterion

  Nyquist plot stays within a circle

- Popov Criterion

  The Popov plot stays on the right of a line

Lyapunov Approach

- Quadratic

  $$V(x) = x^T P x$$

- Qudratic+Integral

  $$V(x) = x^T P x + \int_0^\sigma \phi(y) dy$$

# Common Lyapunov Functions

The following statements are equivalent:

1. The Lur'e system is $[k_1,k_2]$-absolutely stable
2. There is a common Lyapunov function for $A+bk_1c$ and $A+bk_2c$

Molchanov & Pyatnitskiy (1986)

Each of the following function sets provides universal common Lyapunov functions for absolute stability :

- Convex and homogeneous of degree 2
- Polynomials
- Piecewise linear functions
- Piecewise quadratic functions
- Norms

# Extensions

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- Switched Linear Systems

$$\dot{x}(t) \in \{A_1 x(t), \cdots, A_m x(t)\}$$

- Linear Differential Inclusions

$$\dot{x}(t) \in \mathrm{co}\{A_1 x(t), \cdots, A_m x(t)\}$$

The following statements are equivalent:

- The switched linear system is (asymptotically) stable
- The relaxed differential inclusion is (asymptotically) stable
- There is a common (strong) Lyapunov function for

$A_1, \ldots, A_m$

Lin, Sontag, and Wang (1996)

# Remarks

- Lyapunov approach has the full capacity in characterizing the stabilities
- Efficient searching of a Lyapunov function is hard (if not impossible)
- Tractable numerical verification is still unsolved

# Largest Divergence Rate

- The largest divergence rate of a switched linear system is the largest possible rate of divergence with respect to all state trajectories

$$\rho = \sup \lim_{t\to\infty, |x(0)|=1, \sigma} \ln |x(t)| / t$$

- For any switched linear system, the rate is always well-defined and bounded
- The rate is connected to stabilities in an obvious manner

# Matrix Measure

- For any vector norm $|\cdot|$, the induced matrix measure is

$$\mu_{|\cdot|}(A) = \sup \lim_{\tau \to 0+, |x|=1} \frac{|x + \tau A x| - |x|}{\tau}$$

- The measure is well defined, positively homogeneous, convex, and most importantly, satisfies

$$|e^{At}| \le e^{\mu_{|\cdot|}(A)t}, \quad \forall A, t \ge 0$$

# Common Measure of a Matrix Set

- Given a set of matrices $\boldsymbol{A}=\{A_1,\cdots,A_m\}$, and a matrix measure $\mu$, the common measure of the matrix set is
  $$\mu(A_1,\cdots,A_m)=\max\{\mu(A_1),\cdots,\mu(A_m)\}$$
- The measure is well defined, positively homogeneous, convex, and satisfies
  $$|e^{A_{i1}t_1}e^{A_{i2}t_2}\cdots e^{A_{ik}t_k}|\le e^{\mu_{|\cdot|}(A_1,\cdots,A_m)(t_1+\cdots+t_k)}$$
- The least measure is defined to be
  $$\nu(\mathbf{A})=\inf_{|\cdot|}\max\{\mu_{|\cdot|}(A_1),\cdots,\mu_{|\cdot|}(A_m)\}$$
- A measure is extreme if it is equal to the least measure

# Main Result

- For any switched linear system, we have
  $\rho = \nu$

- That is, the largest divergence rate is exactly the least possible common measure of the subsystem matrices set

- For a singleton matrix, the property was established by Zahreddine (2003)

# Outline of the Proof

- The least possible common measure is zero for marginally stable or marginally unstable systems
- For marginally stable or marginally unstable systems, the largest divergence rate is zero
- Other stability cases can be treated by a transition to the marginal case by a normalized transition, and use the linear properties of both the rate and the measure

# Performance Estimation

- For any $n$-dimensional switched linear system, there is a polynomial $P$ with degree less than $n$ such that

$$|x(t)| \leq P(t)e^{\nu t}|x_0|$$

Moreover, $P$ can be chosen to be of degree zero iff the normalized switched system is marginally stable

A marginally unstable system always admit an equivalent form of

$$\bar{A}_i \overset{def}{=} T^{-1}A_iT = \begin{bmatrix} \bar{A}_i^1 & \bar{A}_i^3 \\ 0 & \bar{A}_i^2 \end{bmatrix}$$

where both sub-modes are marginally stable as switched systems

# Stabilities in terms of the Least Measure

- The switched linear system is (exponentially) stable iff the least measure is negative
- The switched linear system is marginally stable iff the least measure is zero and an extreme measure exists
- The switched linear system is marginally unstable iff the least measure is zero and an extreme measure does not exist
- The switched linear system is (exponentially) unstable iff the least measure is positive

# Computing the Least Measure

- Given $\varepsilon > 0$, there exists a natural number *r*>=*n* , a transformation matrix $X_{n \times r}$ of full row rank, and matrices $H_i \in \mathbf{R}^{r \times r}$, such that
  $A_i X = X H_i$
  $\mu_{|\cdot|_1}(H_i) < \nu + \varepsilon$

- Given $\varepsilon > 0$, the least measure could be approximated with the given accuracy by means of a formula based on a sum-of-square with a sufficiently high degree

Inspired by Blanchini (2000)

On-going study

# Summary

- It was found  that the least common matrices measure exactly characterizes the largest possible rate of divergence for the switched linear systems
- This provides a more refined insight into the behavior of switched linear systems than stability
- The result also applies to the linear convex differential inclusions, and Lur’e systems as well

- Approximation of the least measure by means algebraic transformation and recursion

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# *Thank You*

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