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Controllability and Observability for Stochastic Partial Differential Equations

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1. Control problems: from the deterministic to the stochastic

Consider the following controlled system:

$$\begin{cases} \dfrac{d}{dt}y = Ay + Bu, & t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $T > 0$. System (1) is said to be controllable on $(0,T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^1(0,T;\mathbb{R}^m)$ such that $y(T) = y_1$.

Theorem 1: System (1) is controllable on $(0,T) \Leftrightarrow$

$$rank(B, AB, A^2B, \cdots, A^{n-1}B) = n.$$

Put

$$G_T = \int_0^T e^{At} B B^* e^{A^* t} dt.$$

Theorem 2: If system (1) is controllable on $(0, T)$, then $\det G_T \neq 0$. Moreover, for any $y_0, y_1 \in \mathbb{R}^n$, the control

$$u^*(t) = -B^* e^{A^*(T-t)} G_T^{-1} (e^{AT} y_0 - y_1)$$

transfers y_0 to y_1 at time T.

Clearly, if system (1) is controllable on $(0, T)$ (by means of L^1-(in time) controls), then the same controllability can be achieved by using analytic-(in time) controls. We shall see a completely different phenomenon in the simplest stochastic situation.

• The stochastic setting

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by$L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$, with the canonical norm (Similarly, one can define $L^2_{\mathcal{F}}(\Omega; C^k([0,T];H))$ for any positive integer k).

Consider a one-dimensional controlled stochastic differential equation:

$$dx(t) = [bx(t) + u(t)]dt + \sigma dB(t), \tag{2}$$

with b and σ being given constants. We say that system (2) is *exactly controllable* if for any $x_0 \in \mathbb{R}$ and $x_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, there exists a control $u(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^1(0,T;\mathbb{R}))$ such that the corresponding solution $x(\cdot)$ satisfies $x(0) = x_0$ and $x(T) = x_T$.

It is shown by Q. Lü, J. Yong and X. Zhang (JEMS, 2011) that system (2) is exactly controllable at any time $T > 0$ (by means of $L^2_{\mathcal{F}}(\Omega; L^1(0,T;\mathbb{R}))$-controls).

On the other hand, surprisingly, in virtue of a result by S. Peng (Prog. Natur. Sci., 1994), system (2) is NOT exactly controllable if one restricts to use admissible controls $u(\cdot)$ in $L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}))$!

Further, it is shown by Q. Lü, J. Yong and X. Zhang (JEMS, 2011) that system (2) is NOT exactly controllable, either provided that one uses admissible controls $u(\cdot)$ in $L^2_{\mathcal{F}}(\Omega; L^q(0, T; \mathbb{R}))$ for any $q \in (1, \infty]$. This leads to a corrected formulation for the exact controllability of stochastic differential equations, as presented below.

Consider a linear stochastic differential equation:

$$\begin{cases} dy = \big(Ay + Bu\big)dt + \big(Cy + Du\big)dB(t), \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $A, C \in \mathbb{R}^{n\times n}$ and $B, D \in \mathbb{R}^{n\times m}$. Unlike the deterministic case, there exists no universally accepted notion for stochastic controllability so far.

Motivated by the above observation, we introduce the following definition: System (3) is said to be exactly controllable if for any $y_0 \in \mathbb{R}^n$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, there exists a control $u(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^1(0, T; \mathbb{R}^m))$ such that $Du(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$ and the corresponding solution $y(\cdot)$ of (3) satisfies $y(T) = y_T$.

To explain the significance for the study of stochastic controllability, we recall briefly stochastic optimal control problems.

It is well-known that one of the most important results in the determinist optimal control theory (in ODE setting) is the Pontryagin maximum principle, which provides a necessary condition for the optimal control, by means of the adjoint equation of the original controlled system. In the determinist case, it is easy to talk about the adjoint equation.

For example, consider the following controlled (deterministic) evolution system

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + Bu(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{4}$$

where $y(t) \in Y$ is the state variable, $u(t) \in U$ is the control variable, Y and U are called respectively the state space and control space, both of which are suitable Hilbert spaces. A generates an C_0-semigroup on Y, while the control operator B maps U into Y. Then, the adjoint equation of (4) is as follows

$$\begin{cases} \dfrac{d}{dt}z(t) = -A^*z(t), \quad t \in (0,T), \\ z(T) = z^*. \end{cases} \tag{5}$$

However, one cannot simply do the same in the stochastic setting. Indeed, it is impossible to solve the following simplest backward stochastic differential equation (BSDE for short):

$$\begin{cases} dz(t) = 0, & t \in (0, T), \\ z(T) = z^* \end{cases} \tag{6}$$

because the only possible equation to (6) is $z(t) \equiv z^*$, which is not necessary adapted to the filtration $\mathcal{F}_t$. To overcome this difficulty, one has to add an "adjusted" term "$Z(t)dB(t)$" in (6) as follows

$$\begin{cases} dz(t) = Z(t)dB(t), & t \in (0, T), \\ z(T) = z^*. \end{cases} \tag{7}$$

This leads to the appearance of the so-called BSDEs, which is a very active field in the last 30 years, after the fundamental works by J.-M. Bismut (1978), E. Pardoux and S. Peng (1990), etc.

BSDEs and its various invariants play important and fundamental roles in Stochastic Optimal Control (S. Peng (1990), J. Yong and X. Y. Zhou (1999)), Mathematical Finance (N. El Karoui, S. Peng and M. C. Quenez (1997)) and so on.

In some sense, BSDEs is a by-product in the study of stochastic optimal control problem. One can expect a similar GAIN in the study of stochastic controllability problem though it is difficult. I believe that the main concern of the controllability/observability theory in the near future should be that for stochastic differential equations.

Though our definition seems to be a reasonable notion for exact controllability of stochastic differential equations, a complete study on this problem is still under consideration and it does not seem to be easy. Due to this, in what follows we shall relax the requirement of exact controllability, say, to consider the null controllability.

Even in the deterministic setting, one needs to consider only the null controllability (say for the heat equation).

On the other hand, when focusing on null controllability problem, we may proceed in a more general setting, i.e., systems governed by stochastic partial differential equations.

2. Controllability and observability estimate for deterministic PDEs

Exact controllability: For any $y_0, y_1 \in Y$, find (if possible) a $u \in L^2(0,T;U)$ such that the solution of (4) satisfies

$$y(T) = y_1? \tag{8}$$

•From the equation point of view, this is a typical ill-posed problem.

• When $\dim Y = \infty$, one has to relax the exact controllability requirement (8) in many cases. This leads to the notions of approximate controllability, null controllability, partial controllability, etc.

• The controllability theory for finite dimensional linear systems was introduced by R.E. Kalman (1960). Stimulated by Kalman's work, many mathematicians devoted to extend it to more general systems including infinite dimensional systems, and its nonlinear and stochastic counterparts.

• There exists a numerous studies on the controllability of (deterministic) partial differential equations.

Classical works: D. L. Russell (SIAM Rev., 1978); J. L. Lions (SIAM Rev., 1988).

Recent book/survey: J. M. Coron (2007); E. Zuazua (2006).

By means of the Range Inclusion Theorem in Functional Analysis, the null controllability of system (4) can be reduced to the following estimate:

$$|e^{A^*T}z^*|_{Y^*}^2 \leq C \int_0^T |B^* e^{A^*(T-s)} z^*|_{U^*}^2 ds, \qquad \forall\, z^* \in Y^*. \tag{9}$$

Put $z(t) = e^{A^*(T-t)}z^*$. Then $z(\cdot)$ solves equation (5), and inequality (9) can be written as

$$|z(0)|_{Y^*}^2 \leq C \int_0^T |B^* z(s)|_{U^*}^2 ds, \qquad \forall\, z^* \in Y^*. \tag{10}$$

It is notable that there is no control variable in equation (5) any more.

• Observability estimate is a sort of *a priori* estimate. In the deterministic setting, one develops many tools on observability estimate, say spectral approach, Rellich-type multiplier method, moment appoach, microlocal analysis approach, Carleman estimate, etc.

• Similar reduction still works for the controllability problem of stochastic PDEs. However, very little is known for the observability estimate on stochastic PDEs.

• In what follows, I will talk about the recent works (in my group) on observability estimate for stochastic PDEs.

3. Observability estimate for stochastic hyperbolic equations

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a smooth boundary Γ, with Γ_0 a given nonempty open subset of Γ. Put $Q = (0,T) \times G$, $\Sigma = (0,T) \times \Gamma$ and $\Sigma_0 = (0,T) \times \Gamma_0$. Consider the following stochastic wave equation:

$$\begin{cases} dy_t - \Delta y dt = (a_1 y_t + a_2 \cdot \nabla y \\ \qquad\qquad\qquad +a_3 y + f)dt \\ \qquad\qquad\qquad +(a_4 y + g)dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \tag{11}$$

with initial data $(y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$, suitable coefficients a_i ($i = 1,2,3,4$), and source terms f and g. Here, $y_t = \frac{dy}{dt}$.

The solution space for equation (11) is chosen to be the following Banach space (with the canonical norm)

$$\begin{aligned}\mathcal{H}_T = & L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G))) \\ & \bigcap L^2_{\mathcal{F}}(\Omega; C^1([0,T]; L^2(G))).\end{aligned} \tag{12}$$

As in the deterministic case, this is the natural energy space for equation (11).

We are concerned with a partial boundary observability estimate for equation (11), i.e., find (if possible) a constant $\mathcal{C}(a_1, a_2, a_3, a_4) > 0$ such that solutions of equation (11) satisfy

$$
\begin{aligned}
&\big|(y(T), y_t(T))\big|_{L^2(\Omega,\mathcal{F}_T,P;H_0^1(G)\times L^2(G))}\\
&\leq \mathcal{C}(a_1, a_2, a_3, a_4)\Big[\Big|\frac{\partial y}{\partial \nu}\Big|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))} + |f|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\\
&\qquad\qquad + |g|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\Big],\\
&\qquad \forall\ (y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G)).
\end{aligned}
\tag{13}
$$

Here, ν is the unit outward normal vector of G.

The above observability inequality (13) is strongly related to the state observation problem of semilinear stochastic wave equations.

The main difficulty to derive (13): It seems that the usual *multiplier approach*, *spectral approach* and *micro-local analysis-based approach*, which work well for deterministic case, DO NOT work in the present stochastic setting.

Similar results can be established for the stochastic parabolic equation and the stochastic Schrödinger equation, see S. Tang-X. Zhang (2009) and Q. Lü (2010), respectively.

• Review on observability estimate for the deterministic hyperbolic equation

We restrict ourself to the time-invariant case and consider the observability estimate for the equation ($\mathcal{A}$ is an elliptic operator):

$$\begin{cases} w_{tt} + \mathcal{A}w = 0, & \text{in } Q \\ w = 0, & \text{in } \Sigma \\ w(0) = w_0, w_t(0) = w_1, & \text{in } \Omega. \end{cases} \tag{14}$$

That is, for $\emptyset \neq \Sigma_0 \subset \Sigma$,

$$|w_0|^2_{H^1_0(G)} + |w_1|^2_{L^2(G)} \leq C \int_{\Sigma_0} \left| \frac{\partial_{\mathcal{A}} w}{\partial \nu} \right|^2 d\Sigma_0. \tag{15}$$

• When $\mathcal{A} = -\Delta$, $\Sigma_0 = (0,T) \times \Gamma_0$ with Γ_0 to be a suitable subset of Γ, L.F. Ho (1986) established (15) by means of the classical Rellich-type multiplier.

•When $\mathcal{A}$ is a general elliptic operator of second order, and Σ_0 is a general (maybe non-cylinder) subset of Σ, J.L. Lions (SIAM Rev., 1988) posed an open problem on "under which condition, estimate (15) holds?".

• When $\Sigma_0 = (0, T) \times \Gamma_0$ is a cylinder subset of Σ, Lions's problem is almost solved but the general case is a challenging unsolved problem (though the underling equation is linear). The most important result is as follows:

C. Bardos, G. Lebeau $\&$ J. Rauch (1992)'s Geometric Optics Condition.

• Whenever the system is time-variant, one needs to use the global Carleman estimate (a sort of weighted energy approach) to establish the corresponding observability estimate.

• Main result and approach

Similar to the deterministic setting, we shall use a stochastic version of the global Carleman estimate to establish inequality (13) for the stochastic wave equation.

The difficulty to do this is the very fact that, unlike the deterministic situation, equation (11), a stochastic wave equation, is *time-irreversible*. One can not simply mimic the usual Carleman inequality for the deterministic hyperbolic equations.

Instead of the usual smooth weight function, we need to introduce a *singular weight function* to derive the desired Carleman estimate for equation (11).

Fix any $x_0 \in \mathbb{R}^d \setminus \overline{G}$. It is clear that

$$0 < R_0 = \min_{x \in G} |x - x_0| < R_1 = \max_{x \in G} |x - x_0|. \quad (16)$$

Put

$$\Gamma_0 = \Big\{ x \in \Gamma \;\Big|\; (x - x_0) \cdot \nu(x) > 0 \Big\}, \quad (17)$$

where $\nu(x)$ is the unit outward normal vector of G at $x \in \Gamma$.

Assume

$$a_1 \in L^\infty_\mathcal{F}(0,T;L^\infty(G)),\ a_2 \in L^\infty_\mathcal{F}(0,T;L^\infty(G;\mathbb{R}^n)),$$
$$a_3 \in L^\infty_\mathcal{F}(0,T;L^n(G)),\ \ a_4 \in L^\infty_\mathcal{F}(0,T;L^\infty(G)), \tag{18}$$

and

$$f \in L^2_\mathcal{F}(0,T;L^2(G)), g \in L^2_\mathcal{F}(0,T;L^2(G)). \tag{19}$$

In what follows, we use the notation:

$$\begin{aligned}\mathcal{A}(a_1,a_2,a_3,a_4) = \ & |(a_1,a_4)|^2_{L^\infty_\mathcal{F}(0,T;(L^\infty(G))^2)} \\ & +|a_2|^2_{L^\infty_\mathcal{F}(0,T;L^\infty(G;\mathbb{R}^n))} \\ & +|a_3|^2_{L^\infty_\mathcal{F}(0,T;L^n(G))}.\end{aligned} \tag{20}$$

We choose a sufficiently small constant $c \in (0, 1)$ so that (Recall (16) for R_0 and R_1)

$$\frac{(4+5c)R_0^2}{9c} > R_1^2. \tag{21}$$

In the sequel, we take $T(> \ 2R_1)$ sufficiently large such that

$$\frac{4(4+5c)R_0^2}{9c} > c^2T^2 > 4R_1^2. \tag{22}$$

Our observability estimate for equation (11) is stated as follows:

Theorem 3: (X. Zhang, 2008). Let (18)–(19) hold, Γ_0 be given by (17), and T satisfy (22). Then solutions of equation (11) satisfy (13) with

$$\mathcal{C}(a_1, a_2, a_3, a_4) = Ce^{C\mathcal{A}(a_1,a_2,a_3,a_4)}. \tag{23}$$

A deterministic version of (11) reads

$$\begin{cases} w_{tt} - \Delta w = b_1 w_t + b_2 \cdot \nabla w + b_3 w + h & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } G, \end{cases} \tag{24}$$

where $b_1 \in L^\infty(Q)$, $b_2 \in L^\infty(Q;\, \mathbb{R}^n)$, $b_3 \in L^\infty(0,T;\, L^n(G))$, and $h \in L^2(Q)$.

As a special case of Theorems 2.2 and 2.3 in “T. Duyckaerts, X. Zhang and E. Zuazua, 2008” and noting the time-reversibility of equation (24), the following counterpart of Theorem 1 holds: If $T > 2R_1$, then solutions of (24) satisfy

$$
\begin{aligned}
&|(w(T), w_t(T))|_{H_0^1(G)\times L^2(G)} \\
&\leq Ce^{C\left[|b_1|^2_{L^\infty(Q)}+|b_2|^2_{L^\infty(Q;\,\mathbb{R}^n)}+|b_3|^2_{L^\infty(0,T;\,L^n(G))}\right]} \\
&\quad\times\left[\left|\frac{\partial w}{\partial\nu}\right|_{L^2(\Sigma_0)}+|h|_{L^2(Q)}\right], \\
&\quad\forall\,(w_0, w_1)\in H_0^1(G)\times L^2(G).
\end{aligned}
\tag{25}
$$

One can easily replace the left hand side of (25) by $|(w_0, w_1)|_{H_0^1(G)\times L^2(G)}$. However, due to the time-irreversibility of equation (11), in principal one cannot simply do the same in the stochastic setting, i.e., replacing the left hand side of (13) by $|(y_0, y_1)|_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G)\times L^2(G))}$.

Surprisingly, this was done in “Q. Lü, 2010”, exactly in a way of the deterministic setting. This is highly nontrivial by considering the very fact that the stochastic wave equation is time-irreversible.

As mentioned before, in order to prove Theorem 3, one needs to derive a Carleman estimate (with singular weight function) for equation (11). For this purpose, for any (large) $\lambda > 0$ and any $c \in (0, 1)$, set

$$\begin{aligned} &\ell(t,x) = \lambda\Big[|x - x_0|^2 - c\Big(t - \frac{T}{2}\Big)^2\Big], \\ &\theta = e^{\ell}. \end{aligned} \tag{26}$$

Also, for any $\beta > 0$, we set

$$\Theta(t) = \exp\left\{-\frac{\beta}{t(T-t)}\right\}, \quad 0 < t < T. \tag{27}$$

It is easy to see that $\Theta(t)$ decays rapidly to 0 as $t \to 0$ or $t \to T$.

Our Carleman estimate for (11) is as follows:

Theorem 4: (X. Zhang, 2008). Let (18)–(19) hold, Γ_0 be given by (17), and c and T satisfy respectively (21) and (22). Then there is a constant $\beta > 0$ and a constant $\lambda^* > 0$ such that solutions of equation (11) satisfy, for $\lambda \geq \lambda^*$,

$$\begin{aligned} &\lambda \mathbb{E} \int_Q \Theta \theta^2 (y_t^2 + |\nabla y|^2 + \lambda^2 y^2) dxdt \\ &\leq C\mathbb{E}\Big\{ \lambda \int_{\Sigma_0} \Theta \theta^2 \left| \frac{\partial y}{\partial \nu} \right|^2 d\Sigma_0 \\ &\qquad + \int_Q \Theta \theta^2 (f^2 + \lambda g^2) dxdt \Big\}, \\ &\quad \forall\ (y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G)). \end{aligned} \tag{28}$$

4. Future works

- It would be quite interesting to study Carleman and observability estimates for backward stochastic PDEs. To the best of our knowledge, this is a challenging problem, and very little is known in this respect.

- We consider here the simplest case of one dimensional standard Brownian motion. It would be interesting to extend the results in this work to the case of colored (infinite dimensional) noise, or even with both state- and control-dependent noise. But these remain to be done.

• Almost nothing is known on controllability/inverse problems of stochastic PDEs although there are some papers addressing the problem in abstract setting. Nevertheless, recently Q. Lü (2010) proved a significant controllability result for forward stochastic heat equations with one control.

• A sharp condition guaranteeing observability inequality (25) (at least when b_1, b_2, and b_3 are time-invariant) is that the triple (G, Γ_0, T) satisfies the geometric optic condition introduced in "C. Bardos, G. Lebeau, and J. Rauch, 1992". It would be quite interesting to extend this result to the stochastic setting, but this turns out to be a very difficult problem.

Thank You!