Stabilization of the 3 state Moore-Greitzer Model via Integral Quadratic Constraint (IQC) Analysis

Anton Shiriaev

Department of Engineering Cybernetics
Norwegian University of Science and Technology
Trondheim, Norway

Department of Applied Physics and Electronics
Umeå University, Sweden

Stabilization of the 3 state Moore-Greitzer Model via Integral Quadratic Constraint (IQC) Analysis

Outline:

- Moore-Greitzer Model: Surge and Stall Dynamics
- IQC for Moore-Greizer Model
- Steps in Controller Design
- Steps in Analysis
- Concluding Remarks

Moore-Greitzer Model: Surge and Stall Dynamics

The surge subsystem of the 3-state MG-model is

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1}{2}\left[1 - (1+\phi)^3\right] \\ \frac{d}{dt}\psi &= \frac{1}{\beta}\left[\phi - u\right] \end{aligned}$$

where β is a constant; and

- u is a scalar control input;
- ψ is a deviation of average-in-space pressure from steady state;
- ϕ is a deviation of average-in-space flow from steady state.

Moore-Greitzer Model: Surge and Stall Dynamics

The full 3-state MG-model is

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \frac{1}{2}\left[1 - (1+\phi)^3\right] - 3 \cdot R \cdot (1+\phi)$$

$$\frac{d}{dt}\psi = \frac{1}{\beta}\left[\phi - u\right]$$

$$\frac{d}{dt}R = -\sigma \cdot R^2 - \sigma \cdot R \cdot \left[2\phi + \phi^2\right], \quad R(0) \geq 0$$

where σ, β is positive constants; and

– u is a scalar control input;

– ψ is a deviation of average-in-space pressure from steady state;

– ϕ is a deviation of average-in-space flow from steady state;

– R is a stall variable

Moore-Greitzer Model: Surge and Stall Dynamics

Challenges to stabilize the origin of the 3MG-model

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \frac{1}{2}\left[1 - (1+\phi)^3\right] - 3 \cdot R \cdot (1+\phi)$$

$$\frac{d}{dt}\psi = \frac{1}{\beta}\left[\phi - u\right]$$

$$\frac{d}{dt}R = -\sigma \cdot R^2 - \sigma \cdot R \cdot \left[2\phi + \phi^2\right], \quad R(0) \geq 0$$

are

- The linearized dynamics is not stabilizable;
- R cannot be used in feedback;
- Nonlinearity in ϕ-dynamics is known approximately.

Stabilization of the 3 state Moore-Greitzer Model via Integral Quadratic Constraint (IQC) Analysis

Outline:

- Moore-Greitzer Model: Surge and Stall Dynamics
- IQC for Moore-Greizer Model
- Steps in Controller Design
- Steps in Analysis
- Concluding Remarks

IQC for Moore-Greitzer Model

These are properties of nonlinearities

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \underbrace{\frac{1}{2}\left[1 - (1+\phi)^3\right]}_{= w_1(\phi)} \underbrace{-3 \cdot R \cdot (1+\phi)}_{= w_2(\phi, R)}$$

$$\frac{d}{dt}\psi = \frac{1}{\beta}\left[\phi - u\right]$$

IQC for Moore-Greitzer Model

These are properties of nonlinearities

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \frac{1}{2}\underbrace{\left[1 - (1+\phi)^3\right]}_{= w_1(\phi)} - 3 \cdot \underbrace{R \cdot (1+\phi)}_{= w_2(\phi, R)}$$

$$\frac{d}{dt}\psi = \frac{1}{\beta}\left[\phi - u\right]$$

For instance,

$$(-\phi) \cdot w_1(\phi) = \left[1 - (1+\phi)^3\right](-\phi) = \phi^2\left(3 + 3\phi + \phi^2\right)$$

$$\geq \frac{3}{4}\phi^2$$

IQC for Moore-Greitzer Model

These are properties of nonlinearities

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \underbrace{\frac{1}{2}\left[1-(1+\phi)^3\right]}_{= w_1(\phi)} - 3\cdot \underbrace{R\cdot(1+\phi)}_{= w_2(\phi, R)}$$

$$\frac{d}{dt}\psi = \frac{1}{\beta}\left[\phi - u\right]$$

For instance,

$$(-\phi)\cdot w_1(\phi) - \frac{3}{4}\phi^2 = \phi^2\left(\frac{3}{2}+\phi\right)^2 \geq 0$$

or integrals along any solution of a closed loop system

$$\int_0^{t_k}\left[-\phi(t)\cdot w_1(\phi(t)) - \frac{3}{4}\phi^2(t)\right]dt > 0, \qquad 0 < t_1 < t_2 < \ldots$$

Stabilization of the 3 state Moore-Greitzer Model via Integral Quadratic Constraint (IQC) Analysis

Outline:

- Moore-Greitzer Model: Surge and Stall Dynamics
- IQC for Moore-Greizer Model
- Steps in Controller Design
- Steps in Analysis
- Concluding Remarks

Controller Design: Robust Stabilization of Surge

Consider a family of dynamic feedbacks

$$\begin{aligned} u &= \phi - \beta^2 \left\{ \lambda_1 \phi + \lambda_2 \psi + \alpha \left[1 - (1+\phi)^3 \right] + \varepsilon\, z \right\} \\ \dot{z} &= -\phi \end{aligned}$$

for stabilization of the surge subsystem

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1}{2}\left[1 - (1+\phi)^3 \right] \\ \frac{d}{dt}\psi &= \frac{1}{\beta}\left[\phi - u \right] \end{aligned}$$

Controller Design: Robust Stabilization of Surge

Consider a family of dynamic feedbacks

$$\begin{aligned} \boldsymbol{u} &= \phi - \beta^2 \left\{ \lambda_1 \phi + \lambda_2 \psi + \alpha \left[1 - (1+\phi)^3 \right] + \varepsilon\, z \right\} \\ \dot{z} &= -\phi \end{aligned}$$

for stabilization of the surge subsystem

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1}{2}\left[1 - (1+\phi)^3 \right] \\ \frac{d}{dt}\psi &= \frac{1}{\beta}\left[\phi - \boldsymbol{u} \right] \end{aligned}$$

The closed loop system takes the form

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \lambda_1 & \lambda_2 & \varepsilon \\ -1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ \alpha \\ 0 \end{bmatrix}}_{B} w_1(\phi)$$

Stabilization: How to Find Controller Parameters?

The IQC can help us in searching quadratic Lyapunov function:

- Suppose the transfer function $T_{w_1 \to \phi}(s)$ satisfies

$$\mathrm{Re}\left\{T_{w_1 \to \phi}(j\omega)\right\} - \frac{3}{4}\left|T_{w_1 \to \phi}(j\omega)\right|^2 \le 0, \quad \forall \omega$$

- Suppose the matrix $\left(A - \frac{3}{4}B\,[1, 0, 0]\right)$ is Hurwitz

Stabilization: How to Find Controller Parameters?

The IQC can help us in searching quadratic Lyapunov function:

- Suppose the transfer function $T_{w_1\to\phi}(s)$ satisfies

$$\text{Re}\left\{T_{w_1\to\phi}(j\omega)\right\} - \frac{3}{4}\left|T_{w_1\to\phi}(j\omega)\right|^2 \le 0, \quad \forall\omega$$

- Suppose the matrix $\left(A - \frac{3}{4}B\,[1,0,0]\right)$ is Hurwitz

---

Then there is $P = P^T > 0$ such that along any solution

$$x(t) = [\phi(t),\ \psi(t),\ z(t)]^T$$

of the LF: $V(t) = x(t)^T P x(t)$ satisfies the inequality

$$V(t_k) + \int_0^{t_k} \left[-\phi(t)\cdot w_1(\phi(t)) - \frac{3}{4}\phi^2(t)\right] dt \le V(0)$$

Stabilization: How to Find Controller Parameters?

The IQC can help us in searching quadratic Lyapunov function:

- Suppose the transfer function $T_{w_1 \to \phi}(s)$ satisfies

$$\text{Re}\left\{T_{w_1 \to \phi}(j\omega)\right\} - \frac{3}{4}\left|T_{w_1 \to \phi}(j\omega)\right|^2 \le 0, \quad \forall\omega$$

- Suppose the matrix $\left(A - \frac{3}{4}B\,[1, 0, 0]\right)$ is Hurwitz

---

Then there is $P = P^T > 0$ such that along any solution

$$x(t) = [\phi(t),\ \psi(t),\ z(t)]^T$$

of the LF: $V(t) = x(t)^T P x(t)$ satisfies the inequality

$$V(t_k) + \int_0^{t_k} \phi^2(t) \cdot \left[\phi(t) + \tfrac{3}{2}\right]^2 dt \le V(0)$$

Stabilization of the 3 state Moore-Greitzer Model via Integral Quadratic Constraint (IQC) Analysis

Outline:

- Moore-Greitzer Model: Surge and Stall Dynamics
- IQC for Moore-Greizer Model
- Steps in Controller Design
- Steps in Analysis
- Concluding Remarks

Analysis of Closed Loop with Stall Dynamics

The closed loop system with nontrivial stall dynamics

$$
\begin{aligned}
\frac{d}{dt}\phi &= \frac{3}{2}\phi - \psi + \frac{1}{2}w_1(\phi) - 3 \cdot R \cdot (1+\phi) \\
\frac{d}{dt}\psi &= \lambda_1\phi + \lambda_2\psi + \varepsilon z + \alpha\, w_1(\phi) \\
\frac{d}{dt}z &= -\phi \\
\frac{d}{dt}R &= -\sigma \cdot R^2 - \sigma \cdot R \cdot \left[2\phi + \phi^2\right], \quad R(0) \geq 0
\end{aligned}
$$

might be unstable even the surge subsystem is stabilized!

Analysis of Closed Loop with Stall Dynamics

- Local asymptotic stability of the origin can be analyzed by the center manifold arguments (done by Leonid Freidovich, 2009)

Analysis of Closed Loop with Stall Dynamics

- Local asymptotic stability of the origin can be analyzed by the center manifold arguments (done by Leonid Freidovich, 2009)

- Analysis of bounded solutions shows that ω-limit sets coincide with the origin. This conclusion is based on
  - Integrability of stall dynamics
  - Analysis of the first return Poincare map defined by the hypersection: $\frac{d}{dt}R = 0$

Analysis of Closed Loop with Stall Dynamics

- Local asymptotic stability of the origin can be analyzed by the center manifold arguments (done by Leonid Freidovich, 2009)

- Analysis of bounded solutions shows that ω-limit sets coincide with the origin. This conclusion is based on
  - Integrability of stall dynamics
  - Analysis of the first return Poincare map defined by the hypersection: $\frac{d}{dt}R = 0$

- When are all solutions of the closed loop system bounded?

Boundedness of Solutions

We rewrite the closed loop system

$$\begin{aligned}
\frac{d}{dt}\phi &= \frac{3}{2}\phi - \psi + \frac{1}{2}w_1(\phi) - 3 \cdot R \cdot (1+\phi) \\
\frac{d}{dt}\psi &= \lambda_1\phi + \lambda_2\psi + \varepsilon z + \alpha\, w_1(\phi) \\
\frac{d}{dt}z &= -\phi \\
\frac{d}{dt}R &= -\sigma \cdot R^2 - \sigma \cdot R \cdot \left[2\phi + \phi^2\right], \quad R(0) \geq 0
\end{aligned}$$

Boundedness of Solutions

We rewrite the closed loop system

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \lambda_1 & \lambda_2 & \varepsilon \\ -1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ \alpha \\ 0 \end{bmatrix}}_{B_1} w_1(\phi) + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}}_{B_2} w_2(R, \phi)$$

$$\frac{d}{dt} R = -\sigma \cdot R^2 - \sigma \cdot R \cdot \left[2\phi + \phi^2\right], \quad R(0) \geq 0$$

and search for new IQC, if exist, e.g.

$$-\phi \cdot w_1 - \frac{3}{4}\phi^2 - K \cdot |w_2|^2 \geq 0$$

to meet the conditions of KYP lemma.

Boundedness of Solutions (Con'd)

The frequency condition for the IQC valid along solutions

$$-\phi(t) \cdot w_1(t) - \frac{3}{4}\phi(t)^2 - K \cdot |w_2(t)|^2 \geq 0$$

means that the opposite inequality

$$-\mathrm{Re}\left[\tilde{\phi} \cdot \tilde{w}_1\right] - \frac{3}{4}|\tilde{\phi}|^2 - K \cdot |\tilde{w}_2|^2 \leq 0$$

hols for complex numbers related by

$$j\omega \cdot \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \lambda_1 & \lambda_2 & \varepsilon \\ -1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{z} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ \alpha \\ 0 \end{bmatrix}}_{B_1} \tilde{w}_1 + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}}_{B_2} \tilde{w}_2$$

where $\omega \in \mathbb{R}^1$ and w_1, $w_2 \in \mathbb{C}^1$ are any.

Boundedness of Solutions (Con'd)

A, B_1 and B_2 have the particular structure that $K > 0$ exists.

Boundedness of Solutions (Con'd)

A, B_1 and B_2 have the particular structure that $K > 0$ exists.

Hence, even for unbounded solution of the closed loop system

$$x(t) = [\phi(t),\ \psi(t),\ z(t)]^T$$

there is a matrix $P = P^T > 0$, for which the Lyapunov-like function $V(t) = x(t)^T P x(t)$ satisfies

$$V(t_k) + \int_0^{t_k} \phi^2(t) \cdot \left[\phi(t) + \tfrac{3}{2}\right]^2 dt - K \cdot \int_0^{t_k} w_2(t)^2 dt \le V(0)$$

Boundedness of Solutions (Con'd)

A, B_1 and B_2 have the particular structure that $K > 0$ exists.

Hence, even for unbounded solution of the closed loop system

$$x(t) = [\phi(t),\ \psi(t),\ z(t)]^T$$

there is a matrix $P = P^T > 0$, for which the Lyapunov-like function $V(t) = x(t)^T P x(t)$ satisfies the inequality

$$V(t_k) + \\ + \int_0^{t_k} \phi^2(t) \cdot \left[\phi(t) + \tfrac{3}{2}\right]^2 dt - \\ - K \cdot \int_0^{t_k} \left[R(t) \cdot (1 + \phi(t))\right]^2 dt \leq V(0)$$

Boundedness of Solutions (Con'd)

The right-hand side of inequality

$$V(t_k) + \int_0^{t_k} \phi^2(t) \cdot \left[\phi(t) + \tfrac{3}{2}\right]^2 dt - K \cdot \int_0^{t_k} \left[R(t) \cdot (1 + \phi(t))\right]^2 dt \leq V(0)$$

represent the interlay between the main terms

$$\varepsilon_1 \cdot \phi(t_k)^2 + \varepsilon_2 \cdot \left[\int_0^{t_k} \phi(t) dt\right]^2 - K \cdot \int_0^{t_k} R(t)^2 dt$$

Boundedness of Solutions (Con'd)

The right-hand side of inequality

$$V(t_k) + \\ + \int_0^{t_k} \phi^2(t) \cdot \left[\phi(t) + \tfrac{3}{2}\right]^2 dt - \\ - K \cdot \int_0^{t_k} \left[R(t) \cdot (1 + \phi(t))\right]^2 dt \le V(0)$$

represent the interlay between the main terms

$$\varepsilon_1 \cdot \phi(t_k)^2 + \varepsilon_2 \cdot \left[\int_0^{t_k} \phi(t) dt\right]^2 - K \cdot \int_0^{t_k} R(t)^2 dt$$

Integrability of Stall Equation

Given a constant $R(0)$ and a scalar function $\phi(t)$, the corresponding solution $R(t)$ of differential equation

$$\frac{d}{dt} R = -\sigma R^2 - \sigma R \left[2\phi(t) + \phi^2(t)\right] ,$$

if exists, looks as follows

$$R(t) = \frac{R(0) \exp\left(-\sigma \int_0^t \left\{\phi^2(\tau) + 2\phi(\tau)\right\} d\tau\right)}{1 + \sigma R(0) \int_0^t \exp\left(-\sigma \int_0^s \left\{\phi^2(\tau) + 2\phi(\tau)\right\} d\tau\right) ds}$$

Summary

- A certain class of controllers tuned for stabilizing surge dynamics stabilize the full 3MG-model as well;

Summary

- A certain class of controllers tuned for stabilizing surge dynamics stabilize the full 3MG-model as well;
- In analysis we first chosen IQC and then showed that the nonlinearity satisfies it!

Summary

- A certain class of controllers tuned for stabilizing surge dynamics stabilize the full 3MG-model as well;
- In analysis we first chosen IQC and then showed that the nonlinearity satisfies it!
- The arguments are not specific for compressor chracteristic!

Summary

- A certain class of controllers tuned for stabilizing surge dynamics stabilize the full 3MG-model as well;
- In analysis we first chosen IQC and then showed that the nonlinearity satisfies it!
- The arguments are not specific for compressor chracteristic!
- The arguments for output feedback desing are coming.