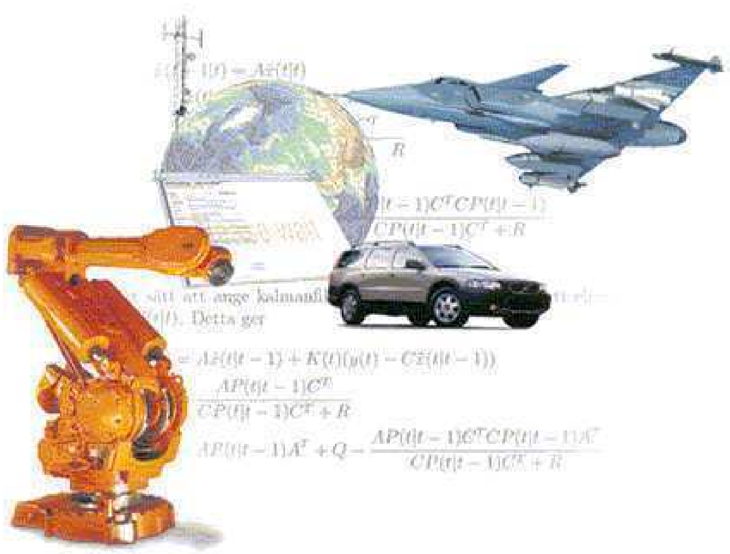
Sum-of-Norm(SON) Regularization in Estimation Problems



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Sparse Approximations (Compressed Sensing)

- Given a matrix A. Approximate it with a sparse matrix (“many” zero elements) $\hat{A}$.
- Make $\|A - \hat{A}\|_2^2$ small while $\|\hat{A}\|_0$ small ($\|x\|_0 = \#$ of nonzero elements in x.)
- various trade-offs controlled by

$$\min_{\hat{A}} \|A - \hat{A}\|_2^2 + \lambda \|\hat{A}\|_0$$

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- Replace the ℓ_0-norm by the ℓ_1-norm!

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Linear System with Occasional Disturbances

$$x(t+1) = A_t x(t) + B_t u(t) + G_t v(t)$$
$$y(t) = C_t x(t) + e(t).$$

Here, e is white measurement noise and v is a process disturbance.

The Process disturbance, v

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But in many applications, v is mostly zero, and strikes only occasionally:

$$v(t) = \delta(t)\eta(t)$$

$$\delta(t) = \begin{cases} 0 & \text{with probability } 1-\mu \\ 1 & \text{with probability } \mu \end{cases}$$

$$\eta(t) \sim N(0, Q)$$

Examples of applications:

- Control: Load disturbance
- Tracking: Sudden maneuvers
- FDI: Additive system faults
- Recursive Identification (x=parameters): model segmentation

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- Branch the KF at each time instant: jump/no jump. Prune/merge trajectories (IMM).

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All methods require some design variables that reflect the trade-off between measurement noise sensitivity and jump alertness.

For one jump, estimate t^* and $v(t^*)$ as parameters.

$$x(t+1) = A_t x(t) + B_t u(t) + G_t v(t)$$
$$y(t) = C_t x(t) + e(t).$$

- If t^* is known it is a simple LS problem to estimate $v(t^*)$. Using the variance of the estimate, the significance of the jump size can be decided in a χ^2 test.
- Find the time of the most significant jump and decide of that is significant enough.
- For detecting several jumps, each detected jump must be accounted for when looking for more.

The Willsky-Jones LS procedure can be written as

Let $$W\big(v(\cdot)\big) = \sum_{t=1}^{N} \big\|\big(y(t) - Cx(t)\big)\big\|^2$$

such that

$$x(t+1) = Ax(t) + Bu(t) + Gv(t);\; x(1) = 0.$$

Solve $$\min_{v(k),k=1,\ldots,N-1} W\big(v(\cdot)\big)$$

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$$\min_{v(k),k=1,\dots,N-1} \sum_{t=1}^{N} \big\|\big(y(t) - Cx(t)\big)\big\|^2 + \lambda\|V\|_0$$

Do the ℓ_1 trick!

This problem is computationally forbidding, so relax the ℓ_0 norm:

$$\min_{v(k),k=1,\ldots,N-1} \sum_{t=1}^{N} \left\| \left(y(t) - Cx(t)\right) \right\|^2 + \lambda \|V\|_1$$

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[StateSON] Compare with Kalman Smoothing:

$$\min_{v(k),k=1,\dots,N-1} \sum_{t=1}^{N} \left\| R_e^{-1/2} \left(y(t) - Cx(t)\right) \right\|^2 + \sum_{t=1}^{N} \|R_v^{-1/2} v(t)\|_2^2$$

Choice of λ

There is a maximal value of λ above which $v(t) \equiv 0$.
It can readily be computed as

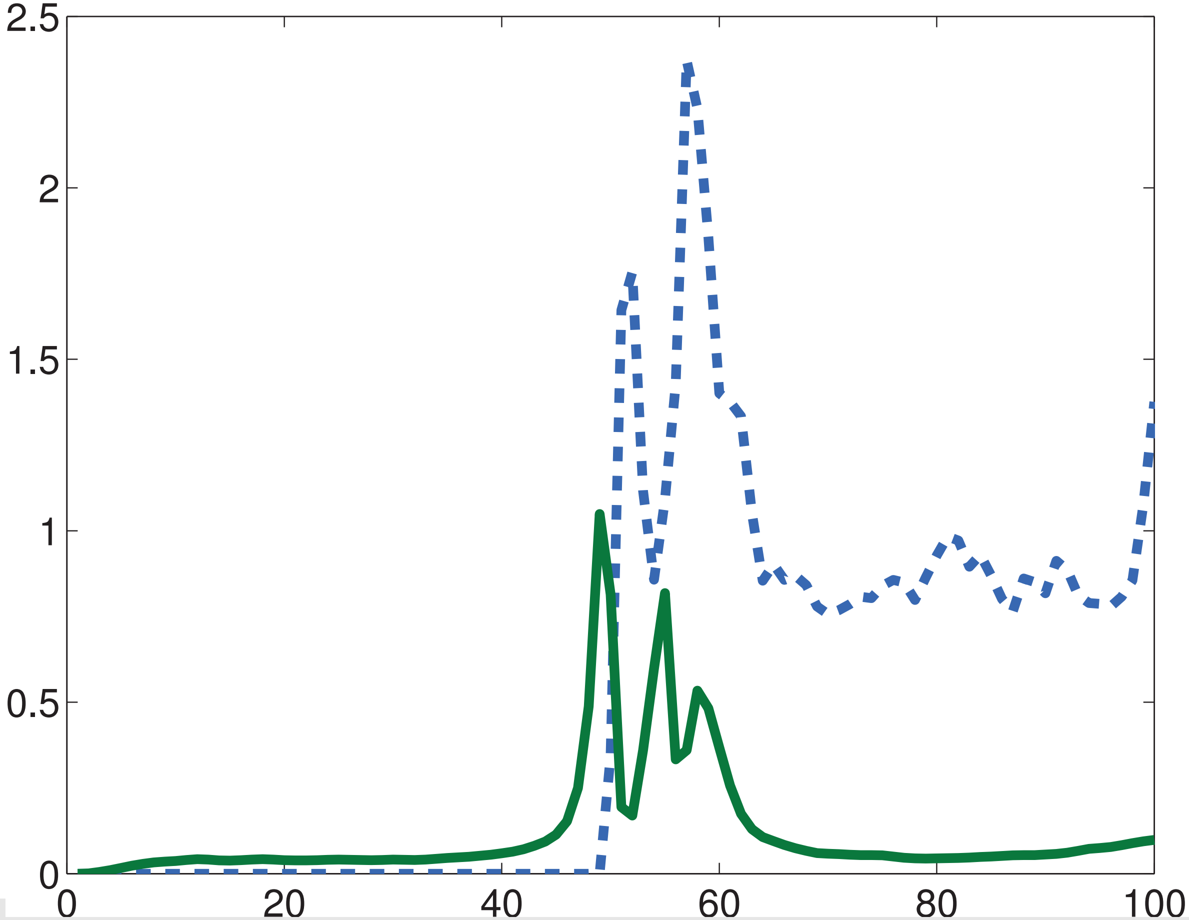
$$\lambda^{\max} = \max_{k=1,\ldots,N-1} \left\| 2 \sum_{t=k+1}^{N} \left(C A^{t-k-1} G \right)^T \varepsilon_t \right\|_2 .$$

where ε are the no-jump residuals from the system.

Scale by assumed SNR.

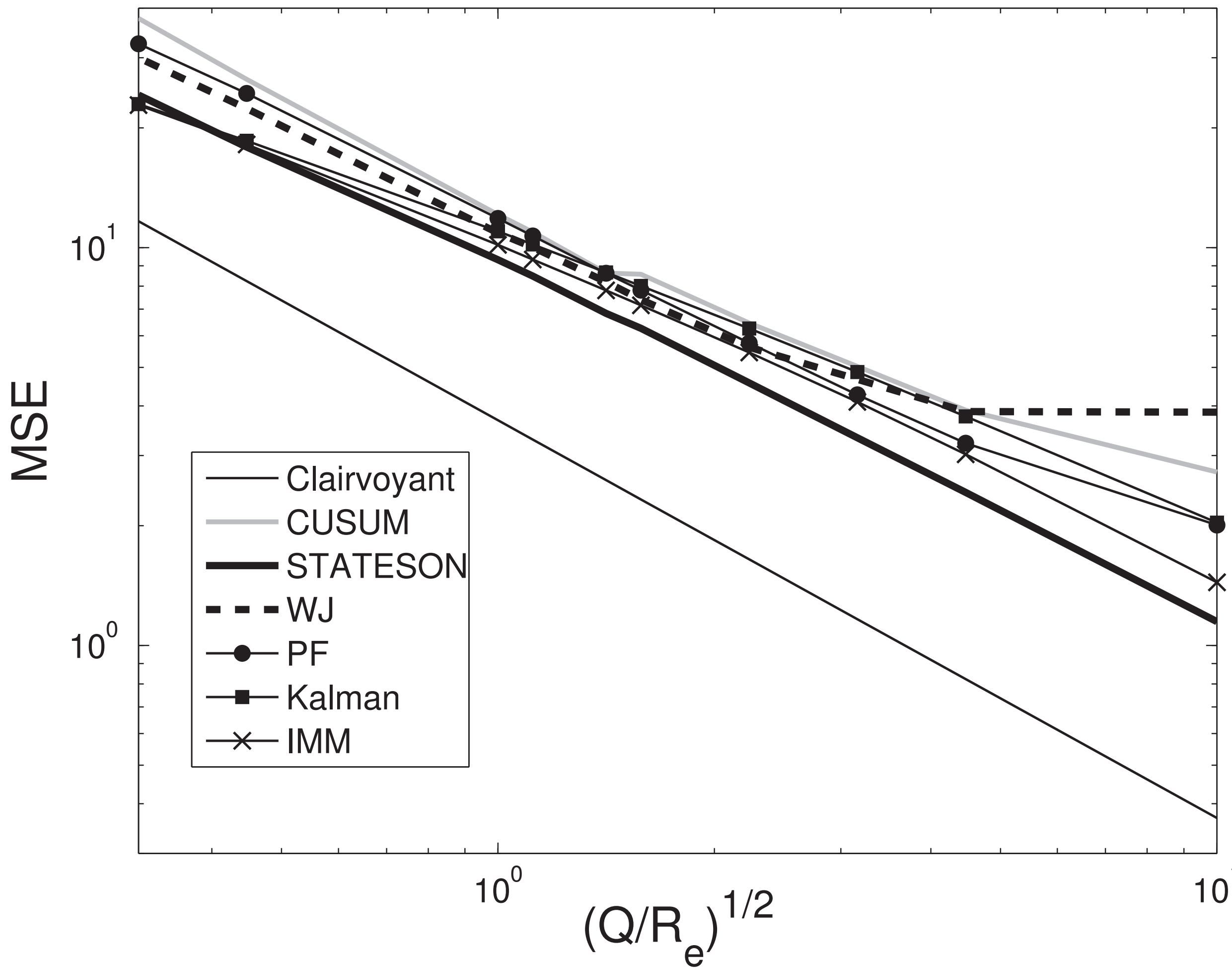
Then use $\lambda = \frac{1}{10}\sqrt{\frac{\|R_e\|}{\|Q\|}}\lambda^{\max}$

DC motor with impulse disturbances at $t = 49, 55$. State RMSE over 500 realizations. Dashed blue: Willsky-Jones, Solid green: StateSON



Varying SNRs

Same system. Jump probability $\mu = 0.015$. Varying SNR: Q = jump size, R_e = measurement noise variance. For each SNR, RMSE averages over 500 MC runs. Many different approaches.



Conclusion

- A ℓ_1 (Sum-of-Norms) relaxation of Willsky-Jones's estimation problem.
- or The standard ML (Kalman smoother) formulation for smoothing with a quadratic regularization term has been studied for the case without squaring the regularization term
- Still Convex with efficient solution methods
- Favors “sparse” solutions
- Good idea for starting values of the regularization parameter λ
- Compares favorably with existing solutions
- Many extensions: Model/signal segmentation, path generation, sensor placement, LPV-modeling, Hybrid models.