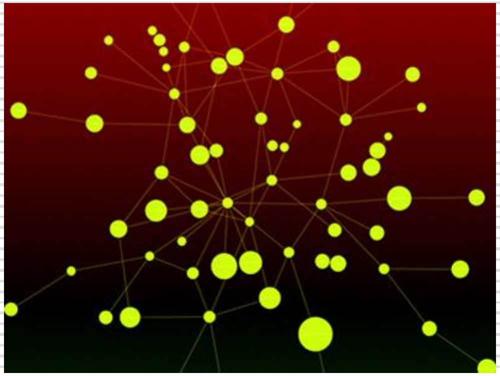
On Pinning Control of Complex Networks



G Ron Chen
City University of Hong Kong

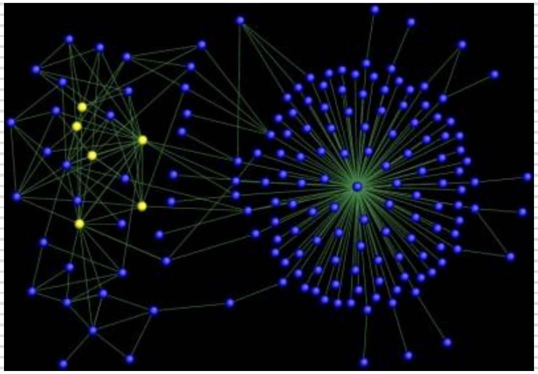
Keywords:

Complex

Network

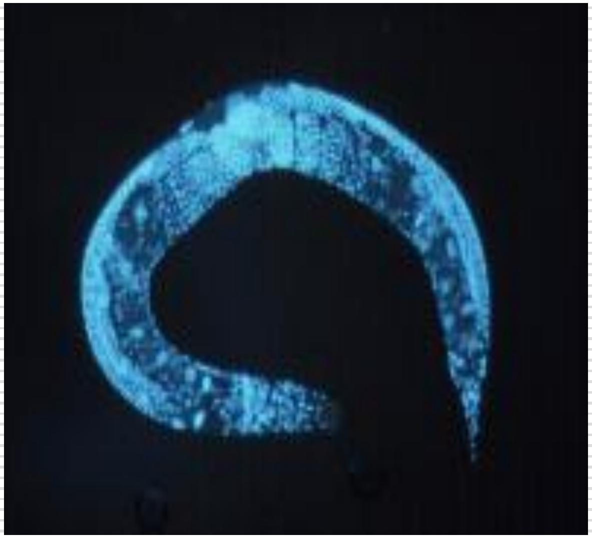
Pinning

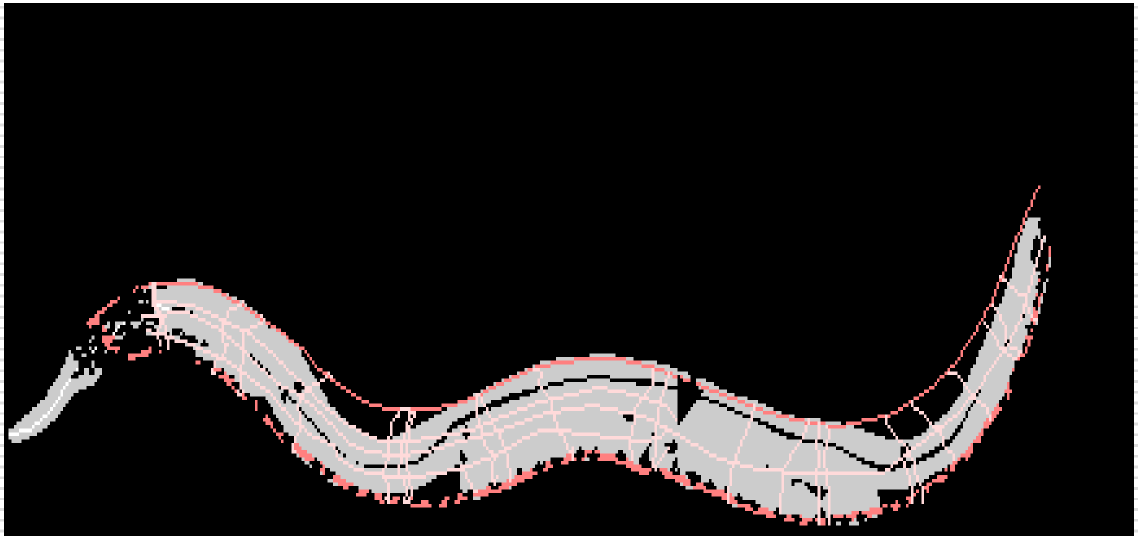
Control



Motivation: An Example

C. elegans





In its neural network:

Neurons: ~ 300   Synapses: ~ 7000

Excerpt

The worm Caenorhabditis elegans has 297 nerve cells. The neurons switch one another on or off, and, making 2345 connections among themselves, They form a network that stretches through the nematode's millimeter-long body.

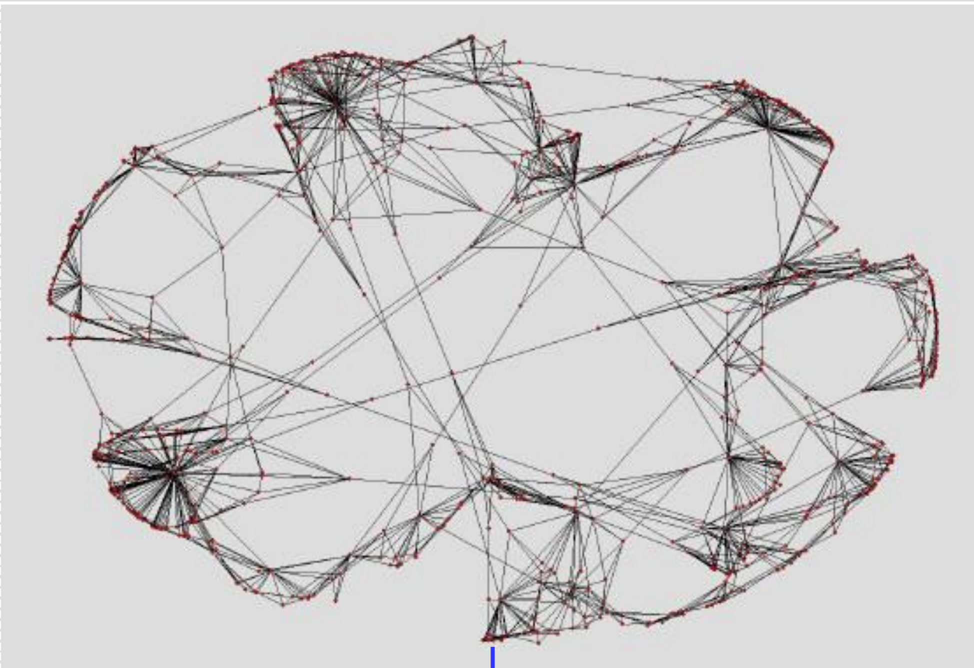
How many neurons would you have to commandeer to control the network with complete precision?

The answer is 49

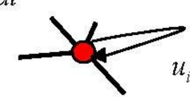
-- Adrian Cho, Science, 13 May 2011, Vol 332, p 777

Here, control = stimuli

Now …



$$\frac{dx_i}{dt} = f(x_i), \qquad x_i \in R^n$$

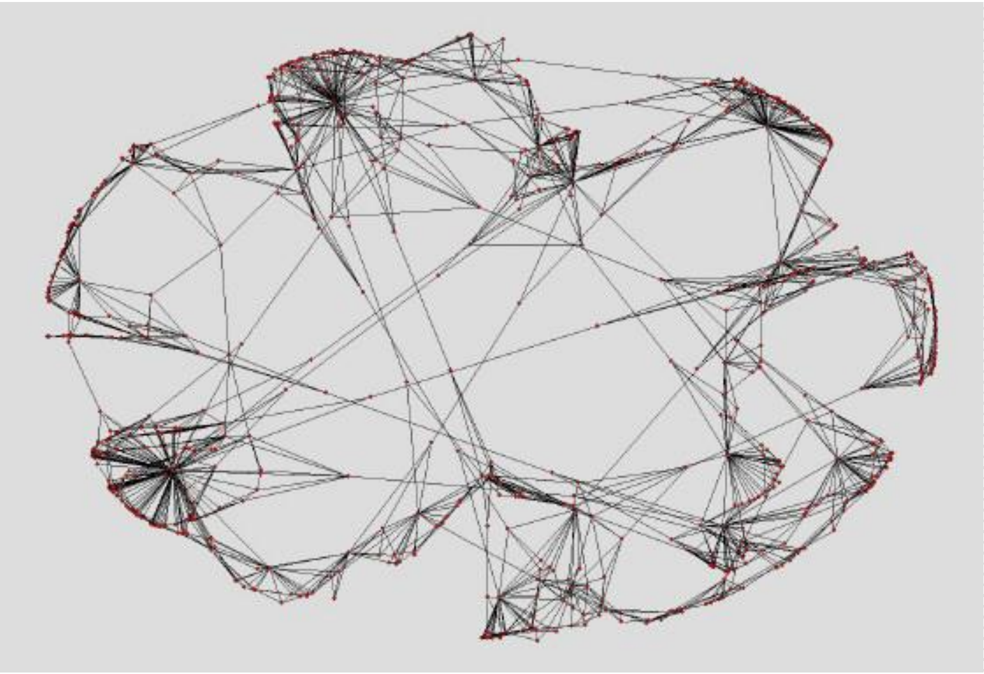


$$u_i = -H_i x_i$$

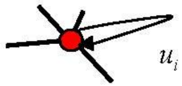
- Given a network of dynamical systems
- Given a specific control objective (e.g. synchronization)
- Assume: a certain class of controllers (e.g., local state-feedback controllers) have been chosen to use

Graph: DataInsight consulting company, Princeton, NJ, USA

Questions:



$$\frac{dx_i}{dt} = f(x_i), \qquad x_i \in R^n$$



$$u_i = -H_i x_i$$

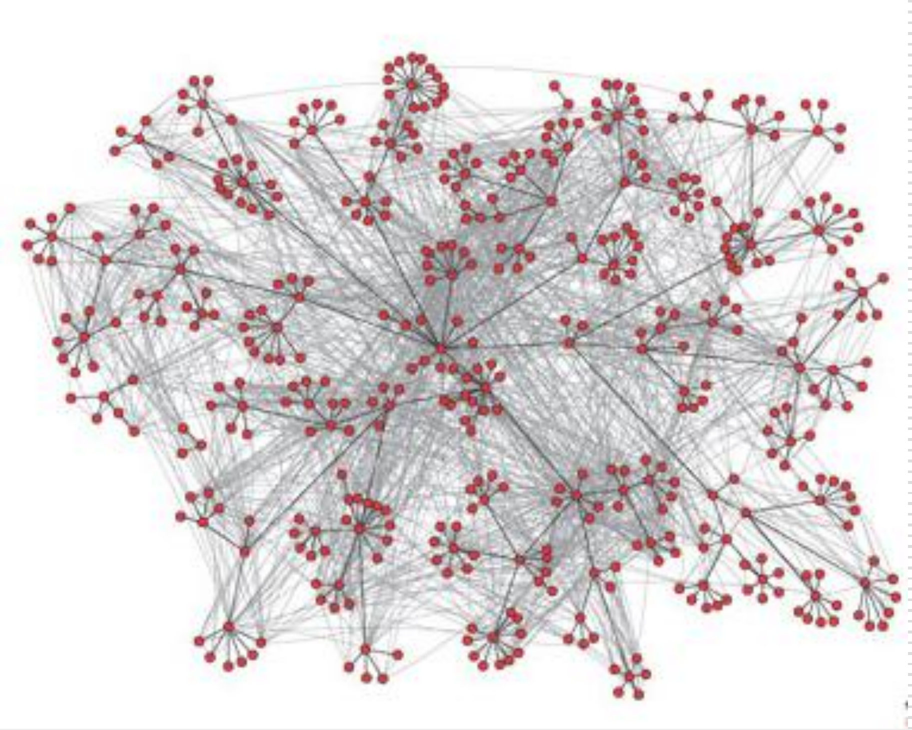
- How many controllers to use?
- Where to put them (which nodes to pin)?

Objective: To achieve cost-effective control (e.g. synchronization) with good performance

Graph: Courtesy of DataInsight, Princeton, NJ, USA

Today's Topics

Only undirected and unweighted networks are discussed

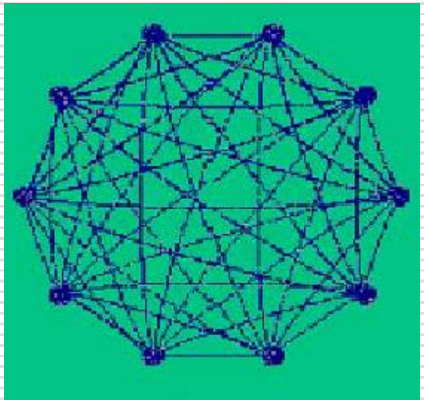


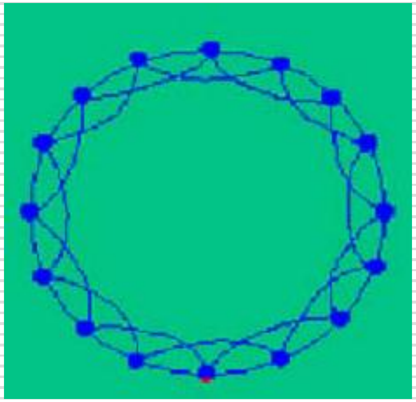
Each node is a higher-dimensional nonlinear dynamical system:

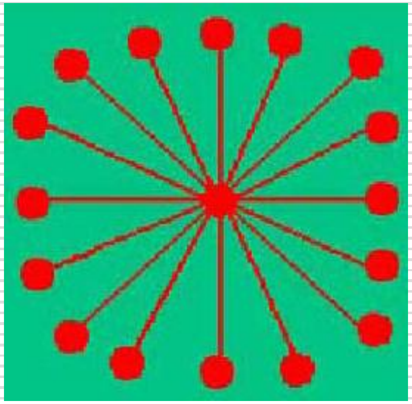
$$\frac{dx_i}{dt} = f(x_i), \quad x \in R^n, \quad i = 1,2,\ldots,N$$

- **Regular networks**
- **Random-graph networks**
- **Small-world networks**
- **Scale-free networks**

Regular Networks







… and so on

If only one controller is applied - pinning location does not matter

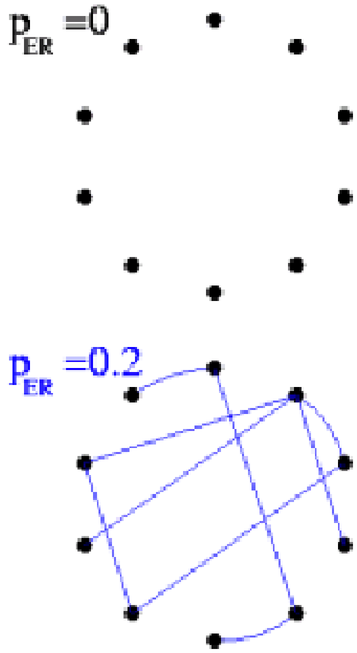
For this one - it does

Random-Graph Networks

Erdős-Rényi

(Publ. Math. Inst. Hung. Acd. Sci. **5,** 17 (1960))

N nodes, each pair of node is connected with probability p



Features:

Connectivity: Poisson distribution

Homogeneity:

- All nodes have similar degrees
- Small average path-length
- Small average clustering coefficient

Non-growing

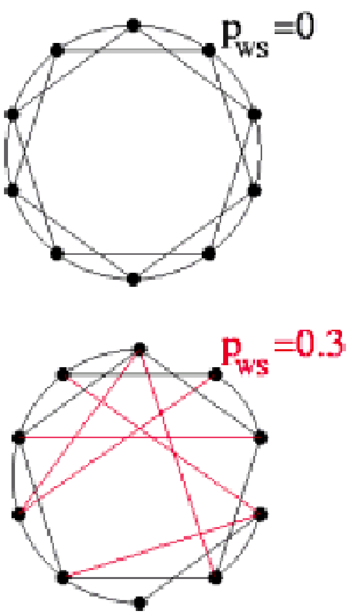
Random pinning – simple, cost-effective, approximately same effects

Selective pinning – costly, need global information, about same effectiveness

Small-World Networks

Watts-Strogatz

(Nature **393**, 440 (1998))



N nodes forms a regular lattice. With probability p, each edge is rewired randomly

Features:

Connectivity: ~ Poisson distribution
Homogeneity:
- **All nodes have similar degrees**
- **Small average path-length**
- **Large average clustering coefficient**

Non-growing

Random pinning – simple, cost-effective, approximately same effects

Selective pinning – costly, need global information, about same effectiveness

Scale-Free Networks

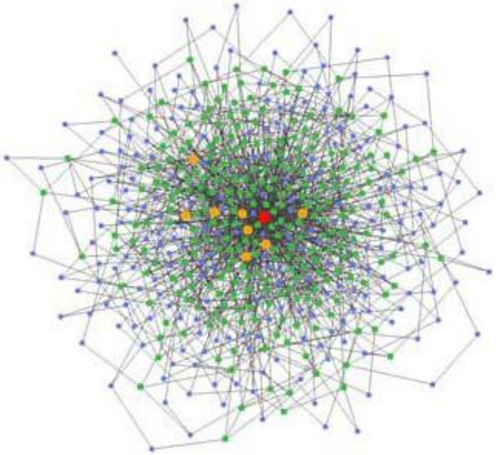
Barabasi-Albert

(Science, **286**: 509 (1999))

Growth with Preferential Attachment:

New incoming node is connecting to each existing node of degree k_i with probability

$$\Pi(k_i) = \frac{k_i}{\sum_l k_l}$$



Features:

Connectivity: Power-law $P(k) \propto k^{-\gamma}$

Non-homogeneity:

- **Few nodes have large degrees**
- **Most nodes have small degrees**

Growing

Random pinning – simple, cost-effective, poor performances

Selective pinning – costly, need global information, very high performances

Network Model

Linearly coupled network:

$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

a general assumption is that *f* (.) is Lipschitz

coupling strength $c > 0$ and

coupling matrices (undirected):

$$A = [a_{ij}]_{N\times N} \qquad H(x_j) = \begin{bmatrix} H_1(x_j) \\ H_2(x_j) \\ \vdots \\ H_n(x_j) \end{bmatrix} \qquad \text{e.g.} \quad H = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{bmatrix}$$

A: If node i connects to node j $(j \neq i)$, then $a_{ij} = a_{ji} = 1$; else, $a_{ij} = a_{ji} = 0$; also, $a_{ii} = 0$

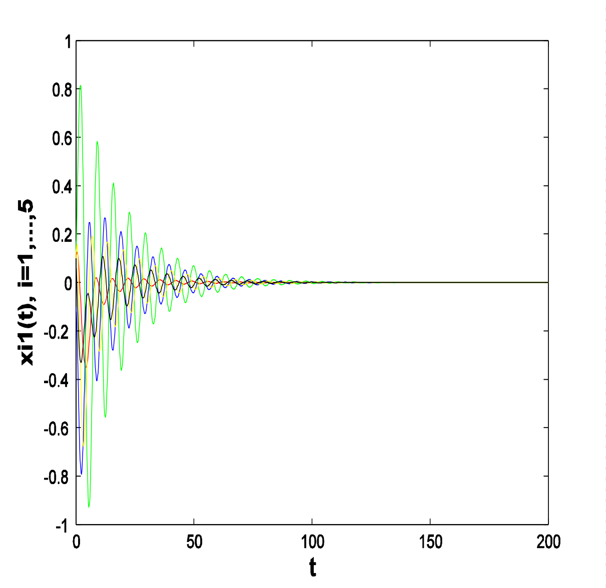
Laplacian matrix: $L = D - A \qquad D = diag\{d_1,\ldots,d_n\} \qquad d_i$ - degree of node i

Network Synchronization

$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

complete state synchronization:

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \qquad i, j = 1,2,\ldots,N$$



Example:

Synchronized to zero equilibrium

What kind of controllers? How many? Where?

$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \leftarrow \quad + u_i \qquad i = 1,2,\ldots,N$$

$$u_i = -Hx_i$$

→

$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij}\delta_{ij} H(x_j - x_i) \qquad i = 1,2,\ldots,N$$

$$\delta_{ij} = \begin{cases} 1 & if \quad to-control \\ 0 & if \quad not-control \end{cases}$$

Q: How many $\delta_{ij} = 1$? Which i and j ?

How many? – One is enough, if …

$$\begin{cases} \frac{dx_1(t)}{dt} = f(x_1(t),t) + c\sum_{j=1}^{m_1} a_{1j}x_j(t) - c\varepsilon(x_1(t)-s(t)) \\ \frac{dx_i(t)}{dt} = f(x_i(t),t) + c\sum_{j=1}^{m_1} a_{ij}x_j(t), \qquad i = 2,\ldots,m_1 \end{cases}$$

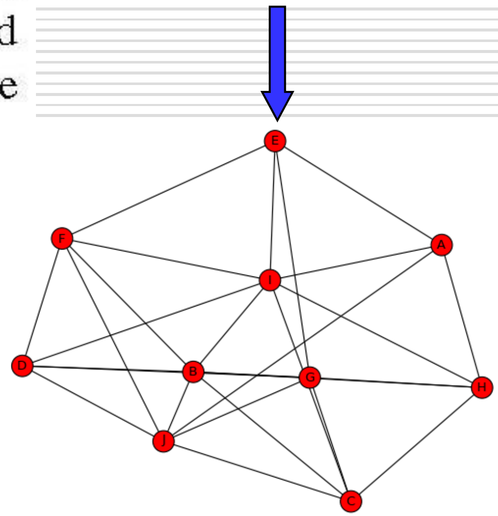
c and ε are large enough

Key Idea

Proposition 1: If $A = (a_{ij})_{i,j=1}^{m}$ is an irreducible matrix with $\text{Rank}(A) = m-1$ and satisfying $a_{ij} = a_{ji} \geq 0$, if $i \neq j$, and $\sum_{j=1}^{m} a_{ij} = 0$, for $i = 1, 2, \ldots, m$. Then, all eigenvalues of the matrix

$$\tilde{A} = \begin{pmatrix} a_{11}-\varepsilon & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

are negative.



T.P. Chen, X.W. Liu, W.L. Lu, IEEE TCAS-I (2007)

Where to apply controllers makes a difference

Recall:

Control Objective: To force the network to synchronize onto the network equilibrium:

$$x_1(t) \to x_2(t) \to \ldots \to x_N(t) \to s, \quad as \quad t \to \infty$$

Here, s is an equilibrium of the network

Pinning Control: (which nodes to pin?)

Only a small fraction of nodes are selected for control:

1. Selective pinning scheme
2. Random pinning scheme

X.F. Wang, G.R. Chen, Physica A (2002), X. Li, X.F. Wang, APWCCS (2003),
X. Li, X.F. Wang, G.R. Chen, IEEE T-CAS (2003,2004),
Z.P. Fan, G.R. Chen, Handbook (2004), CCDC (2005)

Example

Consider a scale-free CNN, which has a zero equilibrium:

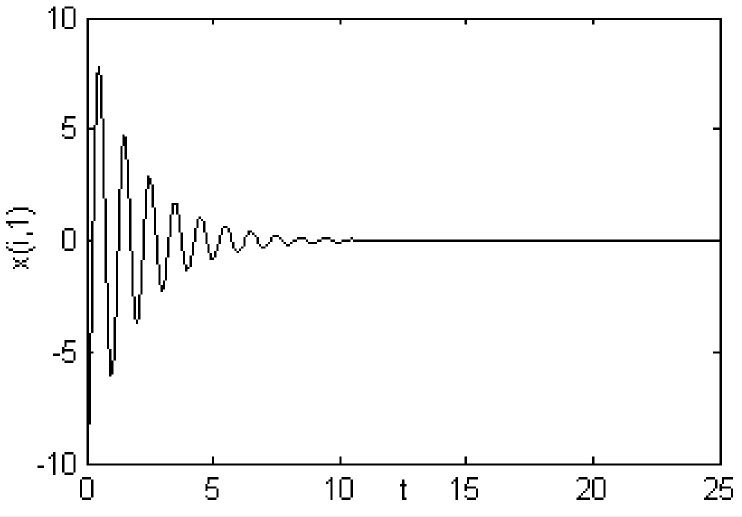
$$\frac{dx_i}{dt} = \begin{pmatrix} \frac{dx_{i1}}{dt} \\ \frac{dx_{i2}}{dt} \\ \frac{dx_{i3}}{dt} \\ \frac{dx_{i4}}{dt} \end{pmatrix} = \begin{pmatrix} -x_{i3} - x_{i4} + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ 2x_{i2} + x_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j2} \\ 14x_{i1} - 14x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j3} \\ 100x_{i1} - 100x_{i4} \\ \quad + 100(|x_{i4}+1| - |x_{i4}-1|) \\ \quad + c\sum_{j=1}^{N} a_{ij} x_{j4} \end{pmatrix} \qquad i = 1, 2, \ldots, N = 60$$

X.F. Wang, G.R. Chen, Physica A (2002)

1 Selective Pinning Control

Here, network size $N = 60$, coupling strength $c = 8.3$ and number of controlled nodes is $m = 15$, by $u_i = -hx_i$

Pin the first 15 largest nodes:



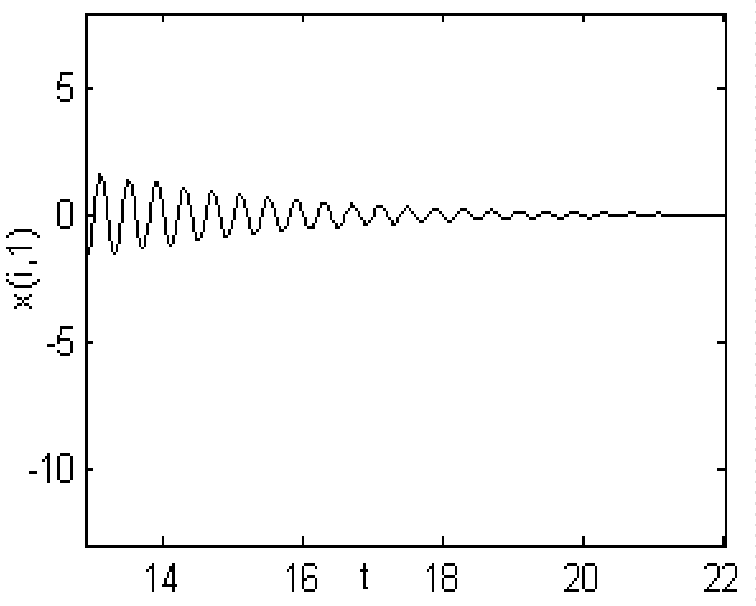
The controlled state x_1

Control gains: $h = 29.8$

Settling time = 10

2 Random Pinning Control

Randomly pin 15 notes. **Comparison:**



The controlled state x_1

1. **Control gain is much larger:**

$$h = 513.4$$

Recall the last one:

$$h = 29.8$$

2. It takes twice longer time to synchronize the network:

Settling time = 20

Recall the last one: 10

X.F. Wang, G.R. Chen, Physica A (2002)

Challenges

Given a network of dynamical systems and control objectives

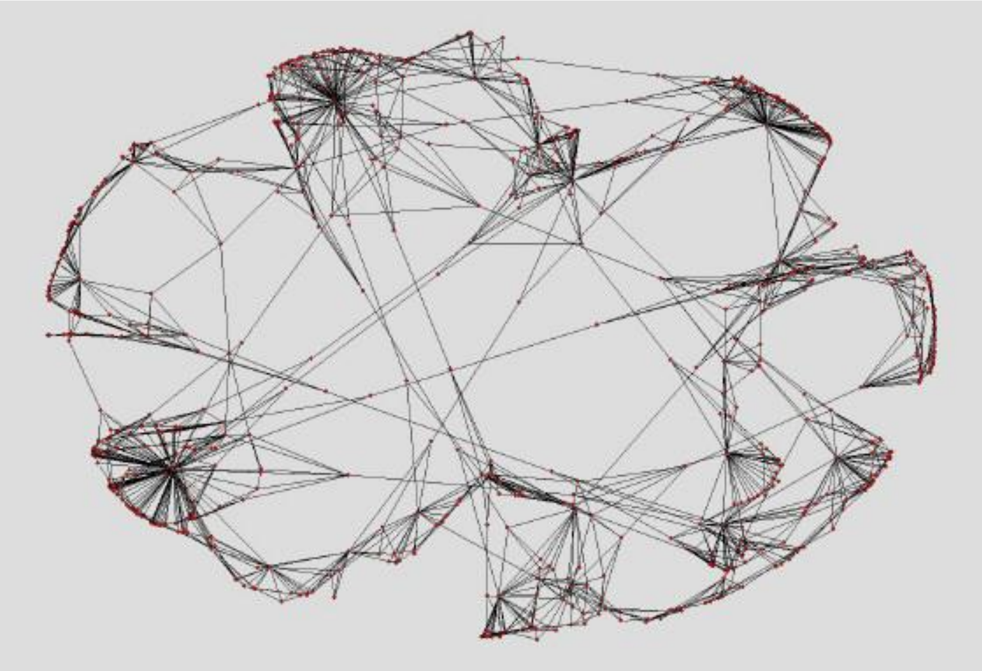
Task 1: Assume a certain class of controllers are chosen to use -
How many controllers to use? Where to apply them?

Task 2: Assume the number of controllers are limited -
What kind of controllers to use? Where to apply them?

Any general theory and methodologies for different types of complex networks of different kinds of dynamical systems?
regular, random-graph, small-world, scale-free, …

-- As of today, however, we have more questions than answers

Thank You！



Q&A