Decomposition of Logical Mappings

with Application to Dynamic-algebraic Boolean Networks

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Outline of Presentation

I. Introduction

☞ Boolean Function

Notations:

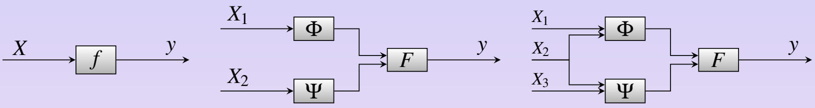
- $\mathcal{D}_k := \{0, \frac{1}{k-1}, \cdots, \frac{k-2}{k-1}, 1\}$, $\mathcal{D} := \mathcal{D}_2$;
- $\delta_k^i = \mathrm{Col}_i(I_k)$;
- $\Delta_k := \{\delta_k^i \mid i = 1, \cdots, k\}$, $\Delta := \Delta_2$;
- $L \in \mathcal{M}_{m\times n}$ is called a logical matrix, if $\mathrm{Col}(L) \subset \Delta_m$, denote it as

$$L = [\delta_m^{i_1}, \cdots, \delta_m^{i_n}] := \delta_m[i_1, \cdots, i_n].$$

Definition 1.1

1. $x \in \mathcal{D}$ is called a Boolean variable;
2. $f : \mathcal{D}^n \to \mathcal{D}$ is called a Boolean function;
3. $F : \mathcal{D}^n \to \mathcal{D}^k$ is called a Boolean mapping.

☞ Decomposition



(a) (b) (c)

- In (a) $X = (x_1, \cdots, x_n)$

$$y = f(X).$$

☞ Decomposition

- Disjoint Decomposition
  In (b) $X = (X_1, X_2)$

$$y = F(\phi(X_1), \psi(X_2)). \tag{1}$$

- Non-Disjoint Decomposition
  In (c) $X = (X_1, X_2, X_3)$

$$y = F(\phi(X_1, X_2), \psi(X_2, X_3)). \tag{2}$$

☞ Semi-tensor Product of Matrices

Definition 1.2

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \operatorname{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \quad (3)$$

☞ Some Basic Comments

- When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$;
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- STP keeps almost all the major properties of the conventional matrix product unchanged.

☞ Algebraic Form of Boolean Function

$$1 \sim \delta_2^1,\ 0 \sim \delta_2^2 \quad \Rightarrow \quad \mathcal{D} \sim \Delta.$$

- Boolean function:

$$f : \mathcal{D}^n \to \mathcal{D} \quad \Rightarrow \quad \Delta^n \to \Delta;$$

- Boolean mapping:

$$F : \mathcal{D}^n \to \mathcal{D}^m \quad \Rightarrow \quad \Delta^n \to \Delta^m.$$

The later function (mapping) is called the vector form.

☞ Algebraic Form

Theorem 1.3

Let $y = f(x_1, \cdots, x_n) : \Delta^n \to \Delta$. Then there exists a unique $M_f \in \mathcal{L}_{2\times 2^n}$ such that

$$y = M_f x, \quad \text{where } x = \ltimes_{i=1}^n x_i. \tag{4}$$

Definition 1.4

The M_f is called the **structure matrix** of f.

☞ Algebraic Form

Theorem 1.5

Let $F : \Delta^n \to \Delta^k$ be defined by

$$y_i = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, k.$$

Then there exists a unique $M_F \in \mathcal{L}_{2^k \times 2^n}$ such that

$$y = M_F x, \tag{5}$$

where

$$x = \ltimes_{i=1}^n x_i; \qquad y = \ltimes_{i=1}^k y_i.$$

Definition 1.6

The M_F is called the **structure matrix** of F.

☞ Structure Matrices of Logical Operators

Table: Structure Matrices of Logical Operators

$\neg$	M_n	$\delta_2[2\ 1]$
$\vee$	M_d	$\delta_2[1\ 1\ 1\ 2]$
$\wedge$	M_c	$\delta_2[1\ 2\ 2\ 2]$
$\rightarrow$	M_i	$\delta_2[1\ 2\ 1\ 1]$
$\leftrightarrow$	M_e	$\delta_2[1\ 2\ 2\ 1]$
$\bar{\vee}$	M_p	$\delta_2[2\ 1\ 1\ 2]$

II. Decomposition of Boolean Functions

Disjoint Decomposition

$$f(x_1, \cdots, x_n) = F(\phi(X_1), \psi(X_2)), \tag{6}$$

where $X_1 = (x_1, \cdots, x_k)$.
Algebraic Form

$$M_f x = M_F M_\phi x^1 M_\psi x^2 = M_F M_\phi \left(I_{2^k} \otimes M_\psi \right) x, \tag{7}$$

where $x^1 = \ltimes_{i=1}^{k} x_i$ and $x^2 = \ltimes_{i=k+1}^{n} x_i$. Hence

$$M_f = M_F M_\phi \left(I_{2^k} \otimes M_\psi \right). \tag{8}$$

Theorem 2.1

f is disjoint decomposable, iff

$$M_f = [\mu_1 M_\psi \ \mu_2 M_\psi \ \cdots \ \mu_{2^k} M_\psi], \tag{9}$$

where

$$M_\psi \in \mathcal{L}_{2\times 2^{n-k}};$$

$\mu_i \in S, \ \forall i$, where S can be:

- **Type 1**:
$$S = S_1 = \{\delta_2[1\ 1],\ \delta_2[2\ 2]\};$$
- **Type 2**:
$$S = S_2 = \{\delta_2[1\ 1],\ \delta_2[1\ 2]\} \text{ or } \{\delta_2[2\ 2],\ \delta_2[1\ 2]\};$$
- **Type 3**:
$$S = S_3 = \{\delta_2[1\ 2],\ \delta_2[2\ 1]\}.$$

Non-Disjoint Decomposition

$$f(x_1,\cdots,x_n) = F(\phi(X_1,X_2),\psi(X_2,X_3)), \tag{10}$$

where $X_1 = (x_1,\cdots,x_{k_1})$, $X_2 = (x_{k_1+1},\cdots,x_{k_2})$,
$X_3 = (x_{k_2+1},\cdots,x_n)$.

Algebraic Form

$$M_f x = M_F M_\phi x^1 x^2 M_\psi x^2 x^3 = M_F M_\phi \left(I_{2^{k_1+k_2}} \otimes M_\psi\right) \left(I_{2^{k_1}} \otimes M_r^{k_2}\right) x. \tag{11}$$

Where $M_r^{k_2}$ is the order reducing matrix, i.e. $[x^2]^2 = M_r^{k_2} x^2$.
Hence

$$M_f = M_F M_\phi \left(I_{2^{k_1+k_2}} \otimes M_\psi\right) \left(I_{2^{k_1}} \otimes M_r^{k_2}\right). \tag{12}$$

Theorem 2.2

$f : \mathcal{D}^n \to \mathcal{D}$ is non-disjoint decomposable, iff

$$M_f = \begin{bmatrix} \mu_{1,1}M_\psi^1 \; \mu_{1,2}M_\psi^2 \; \cdots \; \mu_{1,2^{k_2}}M_\psi^{2_2^k} \\ \mu_{2,1}M_\psi^1 \; \mu_{2,2}M_\psi^2 \; \cdots \; \mu_{2,2^{k_2}}M_\psi^{2_2^k} \\ \vdots \\ \mu_{2^{k_1},1}M_\psi^1 \; \mu_{2^{k_1},2}M_\psi^2 \; \cdots \; \mu_{2^{k_1},2^{k_2}}M_\psi^{2_2^k} \end{bmatrix} \tag{13}$$

where each

$$M_\psi^s \in \mathcal{L}_{2\times 2^{k_3}}, \quad s = 1, \cdots, 2^{k_2};$$

$$\mu_{i,j} \in S, \quad i = 1, \cdots, 2^{k_1}, \; j = 1, \cdots, 2^{k_2},$$

S equals to one of the S_1, S_2, or S_3.

III. Decomposition of Logical Functions

Definition 3.1

Choosing r elements from $\mathcal{L}_{r\times r}$, say,

$$\mathcal{T} = \{T_1, T_2, \cdots, T_r\} \subset \mathcal{L}_{r\times r},$$

$\mathcal{T}$ is called a type.
F is said to have Type $\mathcal{T}$, if the structure matrix of F is

$$M_F = [T_1 \ T_2 \ \cdots \ T_r].$$

Remark: The order of $\{T_i | i = 1, \cdots, r\}$ does not affect the decomposition.

☞ Disjoint Decomposition of r-valued Functions

Theorem 3.2

Let $f : \mathcal{D}_r^n \to \mathcal{D}_r$ be an r-valued logical function with its structure matrix M_f, being split into r^k blocks as

$$M_f = [M_1, M_2, \cdots, M_{r^k}].$$

f is disjoint decomposable, iff there exist

(i) a type $\mathcal{T} = \{T_1, T_2, \cdots, T_r\} \subset \mathcal{L}_{r\times r}$,

(ii) a logical matrix $M_\psi \in \mathcal{L}_{r\times r^{n-k}}$,

such that

$$M_i = T_{s_i} M_\psi, \quad \text{where } T_{s_i} \in \mathcal{T}, \ i = 1, \cdots, r^k. \tag{14}$$

☞ Non-disjoint Decomposition of r-valued Functions

Theorem 3.3

Let $f : \mathcal{D}^n \to \mathcal{D}$ be an r-valued logical function with its structure matrix M_f. f is non-disjoint decomposable, iff

(i) there exists a type $\mathcal{T} \subset \mathcal{L}_{r\times r}$,

(ii) there exist $M_\psi^i \in \mathcal{L}_{r\times r^{k_3}}, \quad i = 1, \cdots, r^{k_2}$,

such that the structure matrix of f can be expressed as

$$M_f = \begin{bmatrix} \mu_{1,1}M_\psi^1 & \mu_{1,2}M_\psi^2 & \cdots & \mu_{1,r^{k_2}}M_\psi^{r^{k_2}} \\ \mu_{2,1}M_\psi^1 & \mu_{2,2}M_\psi^2 & \cdots & \mu_{2,r^{k_2}}M_\psi^{r^{k_2}} \\ \vdots & & & \\ \mu_{r^{k_1},1}M_\psi^1 & \mu_{r^{k_1},2}M_\psi^2 & \cdots & \mu_{r^{k_1},r^{k_2}}M_\psi^{r^{k_2}} \end{bmatrix} \tag{15}$$

where $\mu_{i,j} \in \mathcal{T}$, $\forall i, j$.

☞ Decomposition of mix-valued Functions

Theorem 3.4

1. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \to \mathcal{D}_{r_0}$ with its structure matrix as

$$M_f = [M_1\ M_2\ \cdots\ M_{r_1}], \tag{16}$$

where $M_i \in \mathcal{L}_{r_0 \times r_2}$. f has a decomposed form with respect to $\mathcal{D}_{r_1}$ and $\mathcal{D}_{r_2}$, iff, there exist

(i) a type $\mathcal{T} = \{T_1, T_2, \cdots, T_{r_0}\} \subset \mathcal{L}_{r_0 \times r_0}$,

(ii) a logical matrix $M_\psi \in \mathcal{L}_{r_0 \times r_2}$,

such that

$$M_i = T_{s_i} M_\psi, \quad \text{where } T_{s_i} \in \mathcal{T}, \ \ i = 1, \cdots, r_1. \tag{17}$$

☞ Decomposition of mix-valued Functions

Theorem 3.4(continued)

1. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \times \mathcal{D}_{r_3} \to \mathcal{D}_{r_0}$ be a mix-valued logical function. f is decomposable with respect to $\mathcal{D}_{r_1} \times \mathcal{D}_{r_2}$ and $\mathcal{D}_{r_2} \times \mathcal{D}_{r_3}$, if and only if,

   (i) there exists a type $\mathcal{T} \subset \mathcal{L}_{r_0 \times r_0}$,

   (ii) there exist $M_\psi^i \in \mathcal{L}_{r_0 \times r_3}, \quad i = 1, \cdots, r_2$,

   such that the structure matrix of f can be expressed as

$$M_f = \begin{bmatrix} \mu_{1,1}M_\psi^1 & \mu_{1,2}M_\psi^2 & \cdots & \mu_{1,r_2}M_\psi^{r_2} \\ \mu_{2,1}M_\psi^1 & \mu_{2,2}M_\psi^2 & \cdots & \mu_{2,r_2}M_\psi^{r_2} \\ \vdots & & & \\ \mu_{r_1,1}M_\psi^1 & \mu_{r_1,2}M_\psi^2 & \cdots & \mu_{r_1,r_2}M_\psi^{r_2} \end{bmatrix} \tag{18}$$

   where $\mu_{i,j} \in \mathcal{T}, \quad i = 1, \cdots, r_1,\ j = 1, \cdots, r_2$.

IV. Dynamic-algebraic Boolean Network

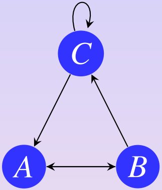


Figure: A Boolean network

☞ Network Dynamics

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t) \end{cases} \tag{19}$$

☞ Dynamics of Boolean Network

$$
\begin{cases}
x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\
\vdots \\
x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \quad x_i \in \mathcal{D},
\end{cases} \tag{20}
$$

where

$$
\mathcal{D} := \{0, 1\}.
$$

☞ Algebraic Form of BN (20)

$$x(t+1) = Lx(t), \tag{21}$$

where $x = \ltimes_{i=1}^{n} x_i$, and $L \in \mathcal{L}_{2^n \times 2^n}$.

☞ Algebraic Form of BN (19)

Example 4.1

Consider Boolean network (6) in Fig. 1. We have

$$L = \delta_8[3\ 7\ 7\ 8\ 1\ 5\ 5\ 6].$$

☞ Dynamic-algebraic BN

$$D - A \;\; BN \; = \; Dynamic\ Part \; + \; Algebraic\ Part.$$

Dynamic Part: X_1

$$x_i(t+1) = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, n-k, \tag{22}$$

Algebraic Part: X_2

$$g_j(x_1, \cdots, x_n) = 1, \quad j = 1, \cdots, k. \tag{23}$$

☞ Solve X_2 out from (23)

Express (23) into the form as:

$$x_j = \phi_j(x_1, \cdots, x_{n-k}), \quad j = n-k+1, \cdots, n. \tag{24}$$

Algebraic form of (23):

$$M_G x^1 x^2 = \delta_{2^k}^1, \tag{25}$$

where $M_G \in \mathcal{L}_{2^k \times 2^n}$.

☞ Constructing Types.

For any positive integer $s > 1$ define a set of matrices, Ξ_i, as

$$\Xi_i = \left\{ E_i \in \mathcal{L}_{s\times s} \middle| \operatorname{Col}_i(E_i) = \delta_s^1;\ \operatorname{Col}_j(E_i) \neq \delta_s^1,\ j \neq i \right\}, \quad i = 1, 2, \cdots \tag{26}$$

Using Ξ_i, we construct a set of types as

$$\mathcal{E}_s := \left[E_1\ E_2\ \cdots\ E_s \right], \quad E_i \in \Xi_i,\ i = 1, 2, \cdots, s. \tag{27}$$

Each type $E \in \mathcal{E}_s$ corresponds to a unique logical mapping $F : \mathcal{D}_s \times \mathcal{D}_s \to \mathcal{D}_s$, which has E as its structure matrix, that is, $M_f = E$.

☞ Key Lemma.

Lemma 5.1

Let $X, Y \in \Delta_s$. $X = Y$, if and only if there exists a $E \in \mathcal{E}_s$ such that

$$EXY = \delta_s^1. \tag{28}$$

☞ Main Result

Theorem 5.2

x_j can be solved as (24) from (23), iff There exists a

$$E = [E_1 \ E_2 \ \cdots \ E_{2^k}] \in \mathcal{E}_{2^k},$$

such that the structure matrix of G can be expressed as

$$M_G = [M_1 \ M_2, \cdots, M_{2^{n-k}}], \tag{29}$$

and

$$M_i \in \{E_1 \ E_2 \ \cdots \ E_{2^k}\}, \quad i = 1, \cdots, 2^{n-k}.$$

☞ An Example

Example 5.3

Consider the follow dynamic-static Boolean network

$$\begin{cases} x_1(t+1) = x_2(t) \to x_4(t) \\ x_2(t+1) = x_1(t) \wedge x_3(t) \\ 1 = (x_3(t)\bar{\nabla}x_4(t)) \leftrightarrow (x_1(t)\bar{\nabla}x_2(t)) \\ 0 = x_4(t)\bar{\nabla}(x_1(t) \vee x_2(t)). \end{cases} \tag{30}$$

We intend to solve x_3 and x_4 out from the last two equations. First, we convert them to

$$\begin{cases} g_1(x_1, x_2, x_3, x_4) := (x_3(t)\bar{\nabla}x_4(t)) \leftrightarrow (x_1(t)\bar{\nabla}x_2(t)) = 1 \\ g_2(x_1, x_2, x_3, x_4) := x_4(t) \leftrightarrow (x_1(t) \vee x_2(t)) = 1 \end{cases} \tag{31}$$

☞ An Example

Example 5.3(continued)

It is easy to calculate that in vector form we have

$$\begin{cases} g_1(x_1, x_2, x_3, x_4) = M_{g_1}x = \delta_2[1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1]x \\ g_2(x_1, x_2, x_3, x_4) = M_{g_2}x = \delta_2[1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 1]x. \end{cases} \tag{32}$$

Then the structure matrix of $G = (g_1, g_2)$ can be easily calculated as

$$M_G = \delta_4[1\ 4\ 3\ 2\ \ 3\ 2\ 1\ 4\ \ 3\ 2\ 1\ 4\ \ 2\ 3\ 4\ 1]. \tag{33}$$

☞ An Example

Example 5.3(continued)

Now we can construct the structure matrix $M_F \in \mathcal{E}_4$ as

$$M_F = \delta_4[1\ 4\ 3\ 2\ \ *\ 1\ *\ *\ 3\ 2\ 1\ 4\ \ 2\ 3\ 4\ 1], \tag{34}$$

where $2 \leq * \leq 4$ can be arbitrary. Comparing (33) with (34) yields that

$$M_\phi = \delta_4[1\ 3\ 3\ 4], \tag{35}$$

which means

$$x_3(t)x_4(t) = \delta_4[1\ 3\ 3\ 4]x_1(t)x_2(t).$$

☞ An Example

Example 5.3(continued)

It follows that $x_3(t)$ and $x_4(t)$ can be solved from (32) uniquely as

$$\begin{cases} x_3(t) = x_1(t) \wedge x_2(t) \\ x_4(t) = x_1(t) \vee x_2(t). \end{cases} \tag{36}$$

plugging (36) into (30) yields

$$\begin{cases} x_1(t+1) = x_2(t) \rightarrow (x_1(t) \vee x_2(t)) \\ x_2(t+1) = x_1(t) \wedge x_2(t). \end{cases} \tag{37}$$

V. Conclusion

1. Two kinds of decompositions of Boolean functions were considered. Necessary and sufficient conditions were obtained.
2. The results have been extended to k-valued and mix-valued logical mappings.
3. Solvability of normal form of dynamic-algebraic logical mapping was considered, and necessary and sufficient conditions were obtained.
4. Semi-tensor product is an useful tool in dealing with Boolean functions.

Thank you!

Question?