LATEX based OH Presentation
Numerical Sensitivity of Linear Matrix Inequalities
Using Shift and Delta Operators
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Numerical Sensitivity of Linear Matrix Inequalities Using Shift and Delta Operators

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Outline

- Delta instead of shift operator for discrete-time dynamic systems
- Numerical sensitivity of Linear Matrix Inequalities
- Ill-conditioned LMI for shorter sampling periods
- Cancellation for shorter sampling periods

Delta operator

Continuous-time system

$$\dot{x} = A_c x$$

Discrete-time system in the shift operator

$$x(t_{k+1}) = A_q x(t_k)$$

$$A_q = e^{hA_c} = \sum_{i=0}^{\infty} \frac{(hA_c)^i}{i!} = I + hA_c + O(h^2) \rightarrow I \quad \text{when} \quad h \rightarrow 0$$

Discrete-time system in the delta operator

$$\delta x(t_k) = \frac{x(t_{k+1}) - x(t_k)}{h} = \frac{A_q - I}{h} x(t_k) \triangleq A_\delta x(t_k)$$

$$A_\delta = \frac{e^{hA_c} - I}{h} \rightarrow A_c \quad \text{when} \quad h \rightarrow 0$$

Two shift operator models

Ordinary shift operator model

$$G_q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix} = \begin{bmatrix} I_n + hA_\delta & hB_\delta \\ C_\delta & D_\delta \end{bmatrix}$$

Signal scaled shift operator model. Input $u_h = \sqrt{h}u$, output $y_h = \sqrt{h}y$

$$G_q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix} = \begin{bmatrix} I_n + hA_\delta & \sqrt{h}B_\delta \\ \sqrt{h}C_\delta & D_\delta \end{bmatrix}$$

The discrete ℓ_2 norm then converges to the continuous $\mathcal{L}_2$ norm

$$\|y_h\|^2 = \sum_{k=0}^{\infty} y_h'(t_k)y_h(t_k) = \sum_{k=0}^{\infty} y'(t_k)y(t_k)\,h$$

Computing the $\mathcal{H}_\infty$ norm

For a stable discrete-time system $\mathcal{G}$, on shift operator form G_q, the $\mathcal{H}_\infty$ norm

$$\|\mathcal{G}\|_\infty = \max_\omega |G_q(e^{i\omega})|$$

With input u and output y, the $\mathcal{H}_\infty$ norm is also given by the induced norm

$$\|\mathcal{G}\|_\infty = \sup_{\|u\|\neq 0} \frac{\|y\|}{\|u\|}$$

Then $\|\mathcal{G}\|_\infty < \gamma$, if and only if there exists a $P = P' > 0$ such that

$$\begin{aligned} P \;>\; & A_q' P A_q + C_q' C_q - (A_q' P B_q + C_q' D_q) \\ & (B_q' P B_q + D_q' D_q - \gamma^2)^{-1} (A_q' P B_q + C_q' D_q)' \end{aligned}$$

and

$$B_q' P B_q + D_q' D_q - \gamma^2 > 0$$

Computing the $\mathcal{H}_\infty$ norm using LMIs

Introducing the notation

$$\begin{aligned} M_{q_{11}}(P) &= A_q' P A_q - P + C_q' C_q \\ M_{q_{12}}(P) &= A_q' P B_q + C_q' D_q \\ M_{q_{22}}(P,\gamma) &= B_q' P B_q + D_q' D_q - \gamma^2 I \end{aligned}$$

we obtain

$$\begin{aligned} M_{q_{11}}(P) - M_{q_{12}}(P) M_{q_{22}}^{-1}(P,\gamma) M_{q_{12}}'(P) &< 0 \\ M_{q_{22}}(P,\gamma) &> 0 \end{aligned}$$

A Schur complement then gives the linear matrix inequality (LMI)

$$M_q(P,\gamma) = \begin{bmatrix} M_{q_{11}}(P) & M_{q_{12}}(P) \\ M_{q_{12}}(P) & M_{q_{22}}(P,\gamma) \end{bmatrix} < 0$$

Minimizing γ satisfying this LMI is a semi-definite program, and the solution gives the $\mathcal{H}_\infty$-norm.

Corresponding LMI on delta operator form

$$M_\delta(P,\gamma) = \begin{bmatrix} M_{\delta_{11}}(P) & M_{\delta_{12}}(P) \\ M_{\delta_{12}}(P) & M_{\delta_{22}}(P,\gamma) \end{bmatrix} < 0$$

$$\begin{aligned} M_{\delta_{11}}(P) &= A'_\delta P + PA_\delta + hA'_\delta PA_\delta + C'_\delta C_\delta \\ M_{\delta_{12}}(P) &= PB_\delta + hA'_\delta PB_\delta + C'_\delta D_\delta \\ M_{\delta_{22}}(P,\gamma) &= hB'_\delta PB_\delta + D'_\delta D_\delta - \gamma^2 I \end{aligned}$$

Cancellation in shift operator LMI

For short sampling periods we have $A_q = I + O(h)$ and

$$M_{q11} = A_q' P A_q - P + C_q' C_q = (I + O(h))' P (I + O(h)) + h C_\delta' C_\delta - P$$
$$= P_h - P + O(h) = O(h)$$

where $P_h \approx P$.

Since $A_q = I + A_\triangle$ where $A_\triangle = h A_\delta$ this cancellation is avoided by replacing $M_{q_{11}} = A_q' P A_q - P + C_q' C_q$ with

$$M_{\triangle_{11}} = A_\triangle' P + P A_\triangle + A_\triangle' P A_\triangle + C_q' C_q$$

Unbounded LMI solution when $h \to 0$

$$M_{q_{11}}(P) = A'_\delta Ph + PhA_\delta + hA'_\delta PhA_\delta + C'_\delta C_\delta = M_{\delta_{11}}(Ph)$$

$$M_{q_{12}}(P) = (I + hA_\delta)PhB_\delta + C'_\delta D_\delta = M_{\delta_{12}}(Ph)$$

$$M_{q_{22}}(P,\gamma) = hB'_\delta PhB_\delta + D'_\delta D_\delta - \gamma^2 I = M_{\delta_{22}}(Ph,\gamma)$$

Hence, we find that

$$M_q(P,\gamma) = M_\delta(\bar{P},\gamma)$$

where $\bar{P} = Ph$.
The solution $P > 0$ to the LMI $M_q(P,\gamma) < 0$ can alternatively be obtained as

$$P = \frac{\bar{P}}{h}$$

Semidefinite programming

LMIs are normally solved as convex optimization problems. Introduce the unknown variables $\xi = [\text{vec}(P)' \;\; \gamma]'$ which gives following semidefinite programming problem

$$\begin{aligned} \min \quad & \gamma \\ \text{subject to} \quad & F(\xi) = \text{diag}(-M_q(P,\gamma),\, P) > 0 \end{aligned}$$

where $F(\xi) \in \Re^{m \times m}$ is symmetric and $m = (2n + n_u)$.
Solved by an interior-point method with the barrier function

$$\phi(\xi) = -\log\det F(\xi)$$

The original criterion γ is then replaced by the approximation

$$f(\xi) = \theta\gamma + \phi(\xi) = \theta\gamma - \log\det F(\xi)$$

where the approximation error is reduced when the parameter θ is increased.

Ill-conditioned problem

$$M_q = \begin{bmatrix} \sqrt{h} I_n & 0 \\ 0 & I_{n_u} \end{bmatrix} M_\delta \begin{bmatrix} \sqrt{h} I_n & 0 \\ 0 & I_{n_u} \end{bmatrix}$$
$$= T_h^{\frac{1}{2}} M_\delta T_h^{\frac{1}{2}} \qquad \Rightarrow \qquad \det M_q = h^n \det M_\delta$$

Since

$$\det F(\xi) = \det \operatorname{diag}(-M_q(P,\gamma),\ P) = (-1)^{n+n_u} \det M_q(P,\gamma) \det P$$
$$= (-1)^{n+n_u} h^n \det M_\delta(P,\gamma) \det P$$

is close to zero independent of γ when h is small, we have an ill-conditioned problem for short sampling periods.
Solution: introduce the scaled LMI problem

$$M_S(P,\gamma) = T_h^{-\frac{1}{2}} M_q(P,\gamma) T_h^{-\frac{1}{2}} = M_\delta(P,\gamma) < 0$$

which gives the same optimal γ as the shift operator LMI, but without the singularity problem for small sampling periods.

Error analysis

Stored relative error using Hadamard (entry-wise) multiplication $\circ$

$$A^{\epsilon} = (\mathbf{1} + \mathcal{E}_A) \circ A$$

Relative error due to subtraction

$$(A - B)^{\epsilon} = (\mathbf{1} + \mathcal{E}_s) \circ \Big((\mathbf{1} + \mathcal{E}_A) \circ A - (\mathbf{1} + \mathcal{E}_B) \circ B \Big)$$

Some manipulations then gives

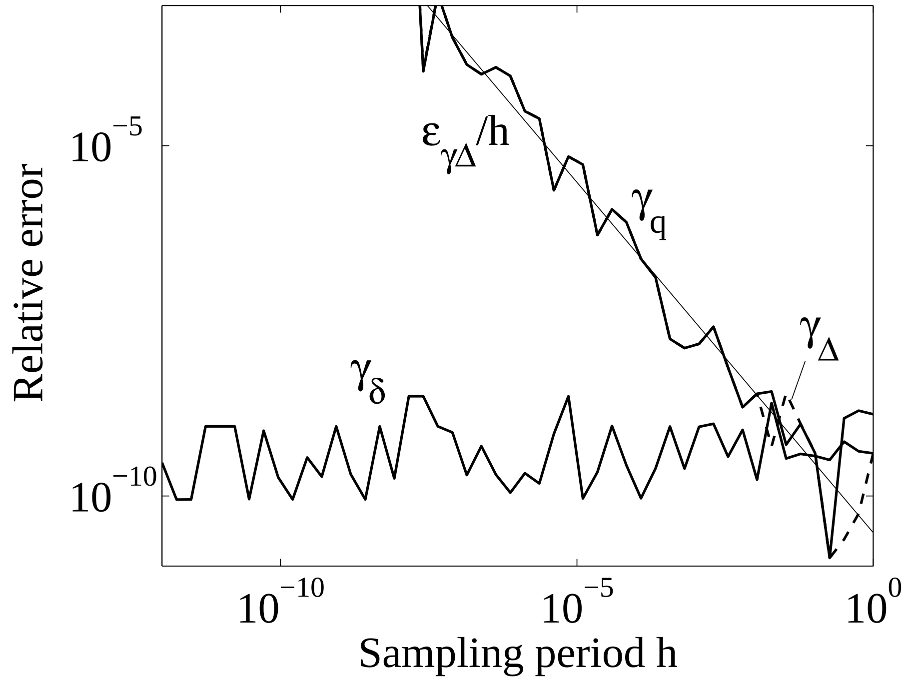
$$M_q^{\epsilon}(P,\gamma) = T_h^{\frac{1}{2}} \left(M_\delta(P,\gamma) + \mathcal{E} \circ M_\delta(P,\gamma) + \frac{1}{h}\text{diag}(\mathcal{E}_P \circ P,\ 0_{n_u \times n_u}) \right) T_h^{\frac{1}{2}}$$

Error estimate

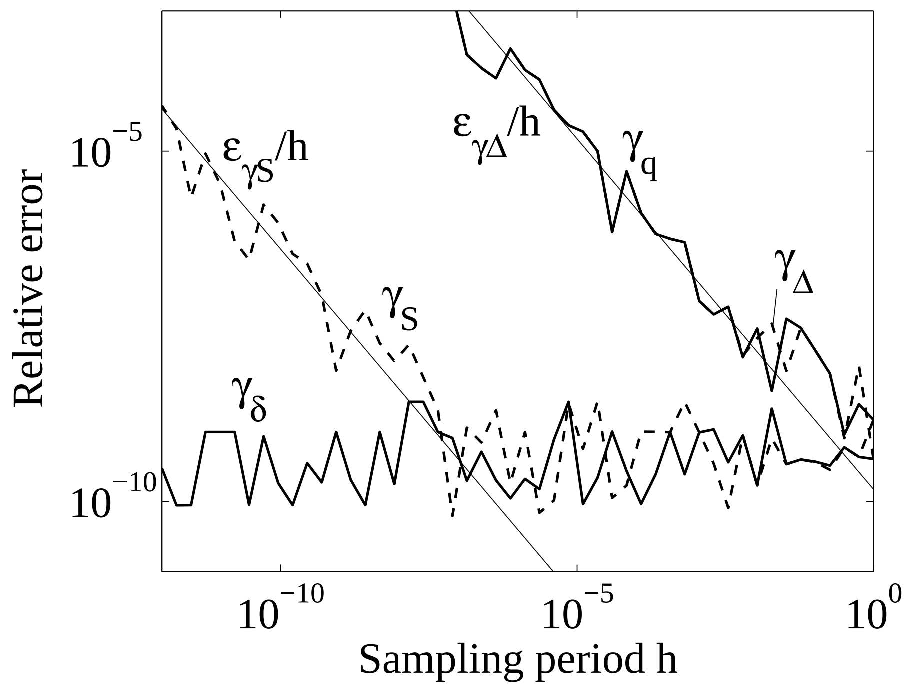
Hence we suggest the following relative error estimate for γ_q

$$e_\gamma = \frac{|\gamma - \gamma_0|}{\gamma_0} \approx \frac{\varepsilon_{\gamma\Delta}}{h} + \frac{\varepsilon_{\gamma S}}{h}$$

$$G_c(s) = \frac{12}{(s+1)(s^2+0.2s+1)(s^2+0.4s+4)}$$



Error estimate for the signal scaled model



Conclusions

- The delta operator is excellent!
- For LMIs the system scaling part of the delta operator seems to be the most severe part from numerical point of view.