# Stability of Sampled-Data Control Loops under Input and Output Jitter

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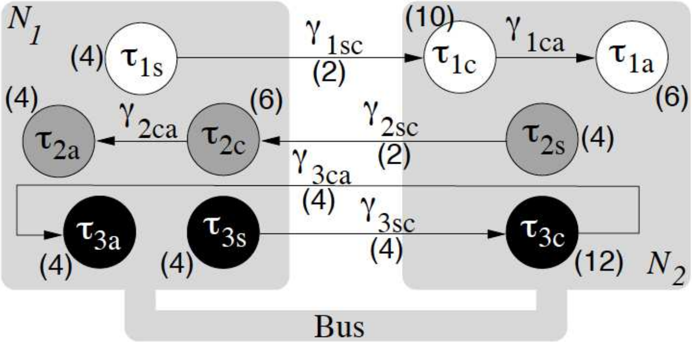
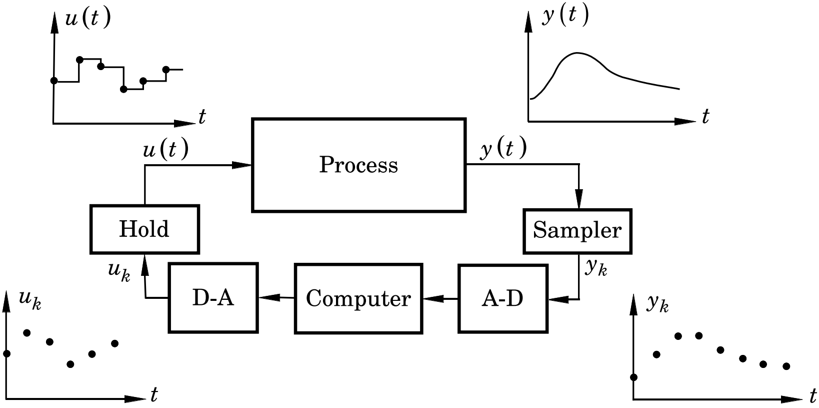
# The Bigger Picture

**Ultimate goal:** Optimal control and scheduling co-synthesis of networked embedded control systems




- Guarantee stability of all control loops
- Minimize performance degradation due to scheduling

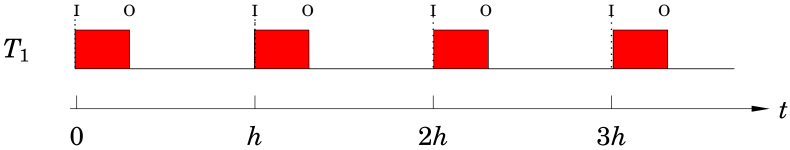
# Textbook Sampled-Data Control




[Åström & Wittenmark, 1997]

# Delay and Jitter due to Scheduling

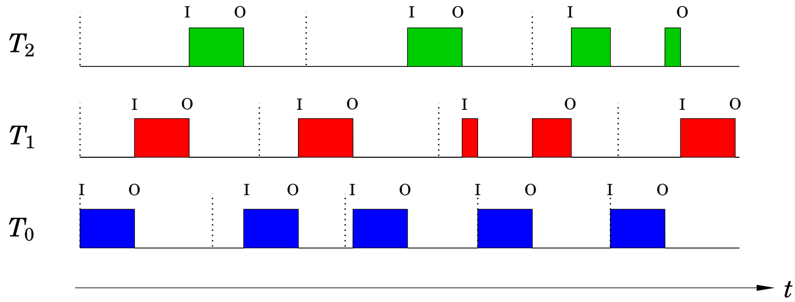
One control task:




- Constant input–output delay
- No jitter
- Textbook analysis applicable

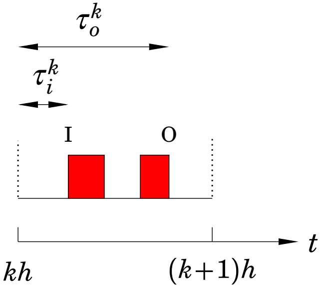
# Delay and Jitter due to Scheduling

Many control tasks (EDF scheduling):




- Both input and output jitter
- Bounds on the jitter can be found by schedulability analysis
- *What about stability and performance?*

# Jitter Definitions




Input jitter: $J_i = \max_k \tau_i^k - \min_k \tau_i^k$

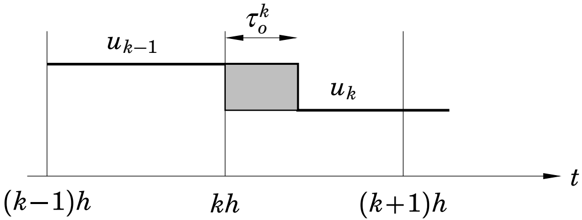
Output jitter: $J_o = \max_k \tau_o^k - \min_k \tau_o^k$

The constant delay part can be included in the plant

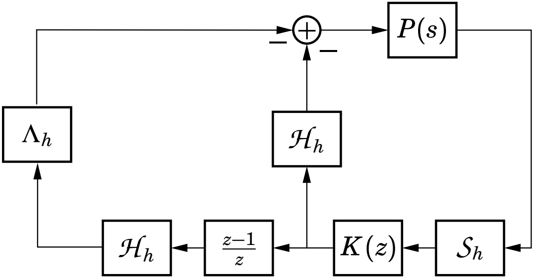
# Analysis of Output Jitter

C.-Y. Kao and B. Lincoln: "Simple Stability Criteria for Systems with Time-Varying Delays," *Automatica* **40**:8, 2004.

- Periodic sampling (no input jitter)
- Linear plant $P(s)$ and controller $K(z)$
- Effect of output jitter modeled as a plant input disturbance:



Assume $J_o \leq h$. Loop transformation:




$\Lambda_h$: time-varying gate function with gain

$$||\Lambda_h||_{L_2} = \sqrt{J_o/h}$$

# Analysis of Output Jitter

**Theorem** [Kao and Lincoln]:

Closed-loop system stable for any $\{\tau_o^k\} \in [0, J_o]$ if

$$\left| \frac{P_{\mathrm{alias}}(\omega) K(e^{i\omega})}{1 + P_{\mathrm{ZOH}}(e^{i\omega}) K(e^{i\omega})} \right| < \frac{1}{\sqrt{J_o/h}\, |e^{i\omega} - 1|} \quad \forall \omega \in [0, \pi]$$

where

$$P_{\mathrm{alias}}(\omega) = \sqrt{\sum_{k=-\infty}^{\infty} \left| P\left( i(\omega + 2\pi k)\frac{1}{h} \right) \right|^2}$$

# Output Jitter – Example

Plant (inverted pendulum):

$$P(s) = \frac{1}{(1 + s10)(1 - s10)}$$

Controller with sampling period $h = 200$ ms:

$$K(z) = \frac{3.36z(z - 0.969)}{z^2 - 1.77z + 0.794}$$

Fixed delay margin: $200$ ms

Jitter margin by Kao and Lincoln: $153$ ms

*How conservative is it?*

# Brute-Force Stability Analysis

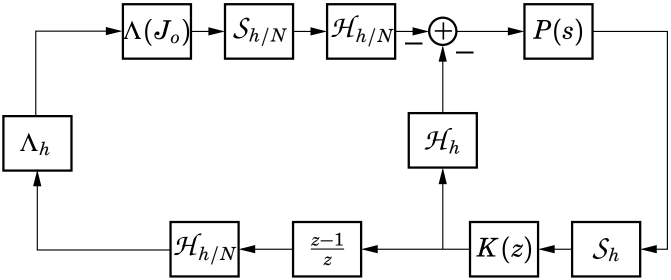
- Assume that the delay can only take on $N$ discrete values between $0$ and $J_o$
- Each delay value gives rise to a closed-loop matrix $\Phi_i$
- $n$-step stability can be asserted by finding a quadratic Lyapunov function for the set of all closed-loop matrices

$$\{\Phi_1, \ldots, \Phi_N\}^n$$

$N = 30$, $n = 2$ gives approximate jitter margin of $189$ ms

# An Improved Analysis

Modified loop transformation:




- Block the transmission of signals with delay outside $[0, J_o]$ using a fixed gate function $\Lambda(J_o)$
- Calculate the gain by fast sampling/fast hold approximation

New jitter margin: 176 ms

# Accounting for Input Jitter

Assume that $P(s)$ has the state-space realization

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

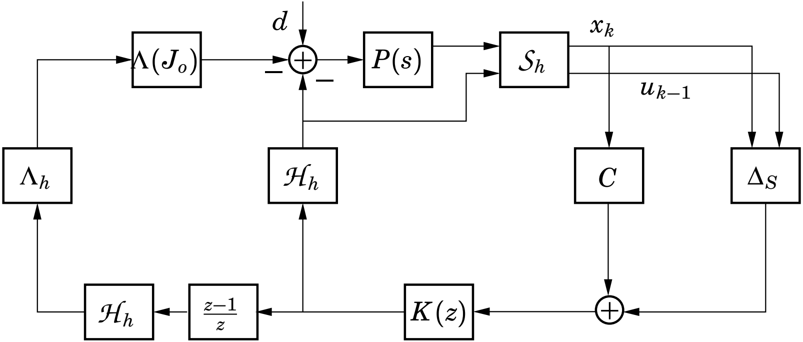
An input delay $0 \leq \tau_i^k \leq J_i$ creates an effective error in the measurement according to

$$\begin{aligned}e_k &= Cx(kh + \tau_i^k) - Cx(kh)\\ &= C\Big(\Phi(\tau_i^k)x_k + \Gamma(\tau_i^k)u_{k-1}\Big) - Cx_k\\ &= \underbrace{\begin{pmatrix} C\big(\Phi(\tau_i^k) - I\big) & C\Gamma(\tau_i^k)\end{pmatrix}}_{\Delta_S}\begin{pmatrix} x_k \\ u_{k-1}\end{pmatrix}\end{aligned}$$

$\Delta_S$ is a time-varying, memory-less MISO operator

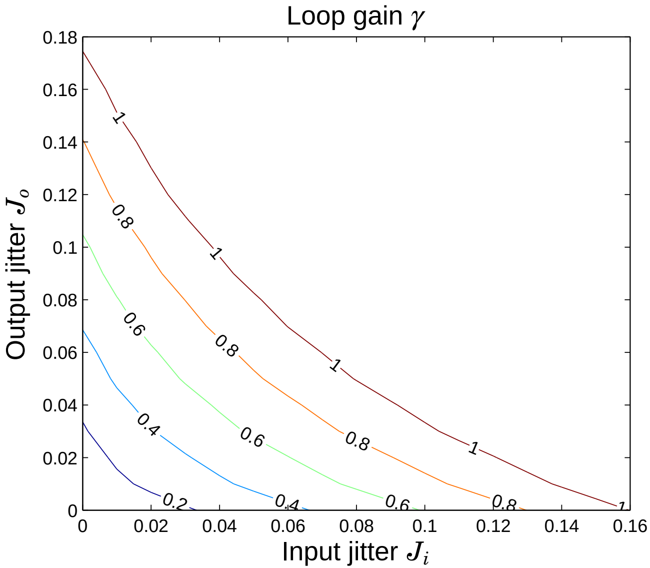
# Accounting for Sampling Jitter

Loop transformation with two uncertainty blocks:




- Small gain theorem

# Example Revisited



# Conclusions

- Multitasking networked control systems give rise to input jitter and output jitter
- Fast stability and performance analysis is needed to carry out efficient co-synthesis of large networked embedded control systems
- Simple extensions of previous results by Kao and Lincoln
  - Reduced conservativeness for sub-sample jitter
  - Input jitter treated as another uncertainty