Scalable Analysis Methods for Sparse Large-scale Systems

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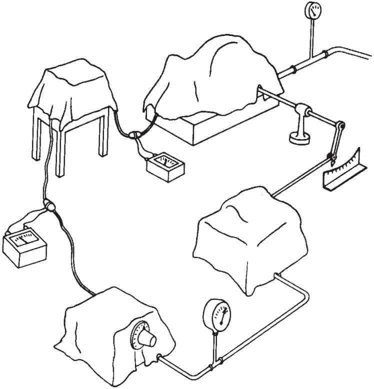
Can Systems be Certified Distributively?



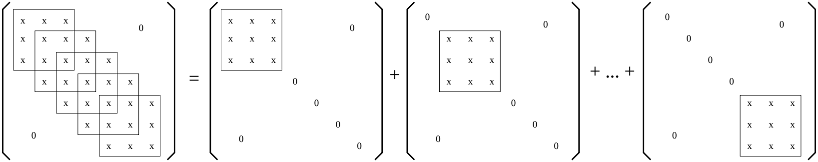
Componentwise performance verification without global model?

Outline

- Introduction
- **Distributed Positive Test for Matrices**
- Distributed Nonconservative System Verification
- A Scalable Robustness Test

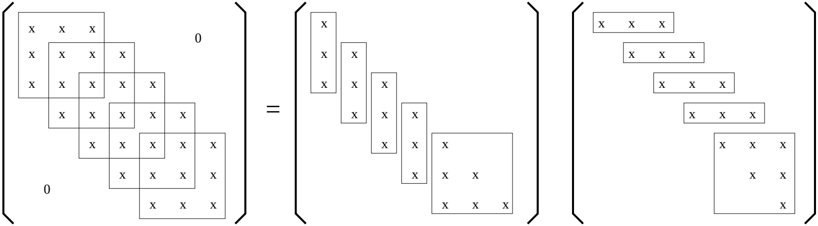
A Matrix Decomposition Theorem

The sparse matrix on the left is positive semi-definite if and only if it can be written as a sum of positive semi-definite matrices with the structure on the right.



Proof idea

The decomposition follows immediately from the band structure of the Cholesky factors:



[Martin and Wilkinson, 1965]

Example

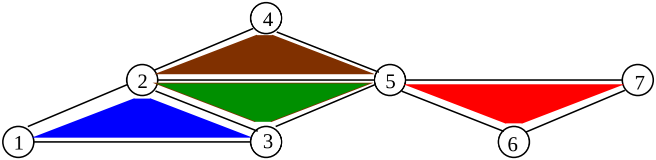
The simplest decomposition is to just split each coefficient equally between the squares where it belong. This could work if the matrix is diagonally dominant:

$$
\begin{pmatrix}
t_{11} & t_{12} & t_{13} & & & & 0 \\
t_{21} & t_{22} & t_{23} & t_{24} & & & \\
t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & & \\
 & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} & \\
 & & t_{53} & t_{54} & t_{55} & t_{56} & t_{57} \\
 & & & t_{64} & t_{65} & t_{66} & t_{67} \\
0 & & & & t_{75} & t_{76} & t_{77}
\end{pmatrix}
=
\begin{pmatrix}
t_{11} & t_{12} & t_{13} & & & & 0 \\
t_{21} & \frac{t_{22}}{2} & \frac{t_{23}}{2} & & & & \\
t_{31} & \frac{t_{32}}{2} & \frac{t_{33}}{3} & & & & \\
 & & & 0 & & & \\
 & & & & 0 & & \\
 & & & & & 0 & \\
0 & & & & & & 0
\end{pmatrix}
+ \dots +
\begin{pmatrix}
0 & & & & & & 0 \\
 & 0 & & & & & \\
 & & 0 & & & & \\
 & & & 0 & & & \\
 & & & & \frac{t_{55}}{3} & \frac{t_{56}}{2} & t_{57} \\
 & & & & \frac{t_{65}}{2} & \frac{t_{66}}{2} & t_{67} \\
0 & & & & t_{75} & t_{76} & t_{77}
\end{pmatrix}
$$

Generalization

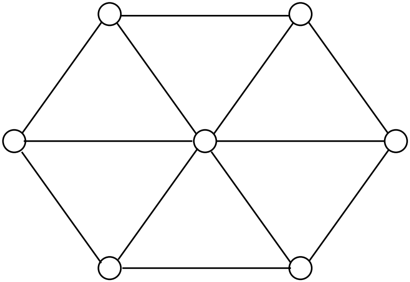
Cholesky factors inherit the sparsity structure of the symmetric matrix if and only if the sparsity pattern corresponds to a “chordal” graph.

$$
\begin{pmatrix}
* & * & * & & & & \\
* & * & * & * & * & & \\
* & * & * & & * & & \\
 & * & & * & * & & \\
 & * & * & * & * & * & * \\
 & & & & * & * & * \\
 & & & & * & * & *
\end{pmatrix}
=
\begin{pmatrix}
* & * & * & & & & \\
* & * & * & & & & \\
* & * & * & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & &
\end{pmatrix}
+
\begin{pmatrix}
 & & & & & & \\
 & * & * & & * & & \\
 & * & * & & * & & \\
 & & & & & & \\
 & * & * & & * & & \\
 & & & & & & \\
 & & & & & &
\end{pmatrix}
+
\begin{pmatrix}
 & & & & & & \\
 & * & & * & * & & \\
 & & & & & & \\
 & * & & * & * & & \\
 & * & & * & * & & \\
 & & & & & & \\
 & & & & & &
\end{pmatrix}
+
\begin{pmatrix}
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & * & * & * \\
 & & & & * & * & * \\
 & & & & * & * & *
\end{pmatrix}
$$



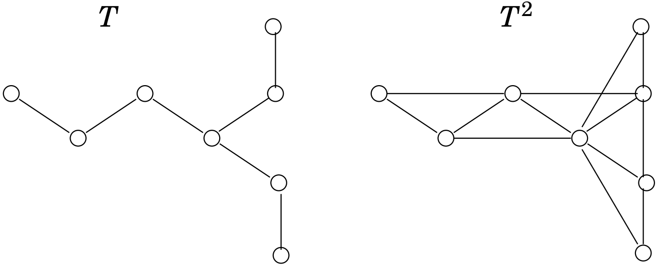
[Blair & Peyton, An introduction to chordal graphs and clique trees, 1992]

Example: Non-chordal graph



Example: Chordal graphs

If T is a tree, then T^k is chordal for every $k \geq 1$.



A Theorem on Positive Extensions

A matrix with entries specified according to a chordal graph has a positive definite completion if and only if all fully specified principal minors are positive definite. [Grone, et.al, 1984]

$$\begin{pmatrix} 3 & 2 & 1 & * & * & * & * \\ 2 & 4 & 2 & 1 & * & * & * \\ 1 & 2 & 4 & 1 & 1 & * & * \\ * & 1 & 1 & 3 & 1 & 1 & * \\ * & * & 1 & 1 & 5 & 2 & 1 \\ * & * & * & 1 & 2 & 4 & 1 \\ * & * & * & * & 1 & 1 & 3 \end{pmatrix}$$

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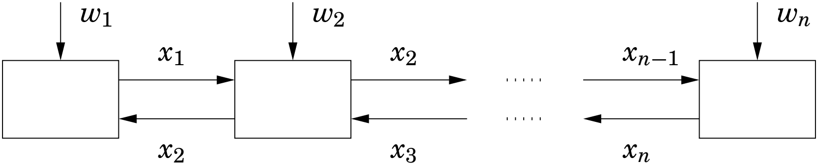
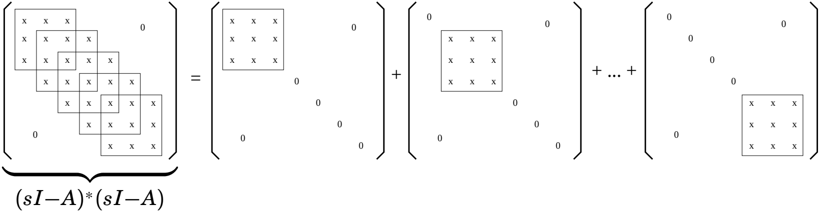
A System with Tridiagonal Structure



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & 0 \\ a_{21} & a_{22} & \ddots & \\ & \ddots & & a_{(n-1)n} \\ 0 & & a_{n(n-1)} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_n(t) \end{bmatrix}$$

A Sparse Stability Test

For the sparse matrix A, let the left hand side illustrate the structure of $(sI - A)^*(sI - A)$. Then the matrix is stable if and only if the right hand side split can be done with all squares positive definite for s in the right half plane.



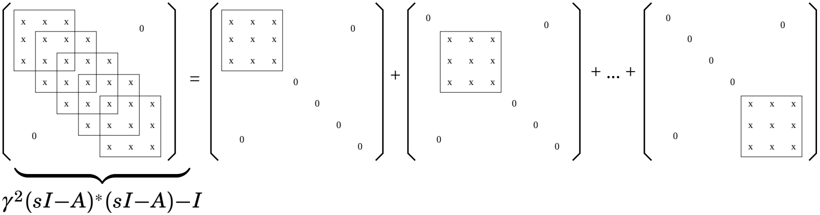
Hence global stability can always be verified by local tests!

A Sparse Gain Bound

Solutions to $\dot{x}(t) = Ax(t) + w(t)$, $x(0) = 0$ satisfy

$$\int_0^T |x(t)|^2 dt \leq \gamma^2 \int_0^T |w(t)|^2 dt$$

if and only if



where the terms on the right hand side are positive definite for s in the right half plane.

A Sparse Passivity Test

Suppose

$$\dot{x} = Ax + Bx + w \qquad\qquad x(0) = 0$$
$$y = Cx$$

Then

$$\int_0^T \left(\gamma^2 u(t)y(t) + |w(t)|^2\right) dt \geq 0 \qquad \text{for all } u, w, T$$

if and only if the matrix

$$\begin{bmatrix} (sI - A)^*(sI - A) & \gamma^2 C^T - (sI - A)^* B \\ \gamma^2 C - B^*(sI - A) & B^T B \end{bmatrix}$$

is positive semi-definite for $\mathrm{Re}\, s \geq 0$.

Passivity can be tested componentwise without conservatism!

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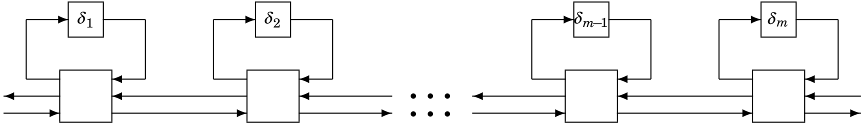
Robustness Analysis for Chained System

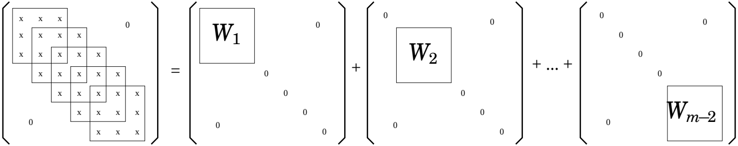


Many robustness analysis problems can be reduced to proving that $(I - \Delta(s)G(s))^{-1}$ is stable for $\Delta = \operatorname{diag}\{\delta_1, \ldots, \delta_m\}$ with $|\delta_i(i\omega)| \leq 1$. This can be done by finding $X(\omega) = \operatorname{diag}\{x_1(\omega), \ldots, x_m(\omega)\} \succ 0$ with $X(\omega) \succ G(i\omega)X(\omega)G(i\omega)^*$ where

$$G(i\omega) = \begin{bmatrix} g_1 & h_1 & 0 & 0 & 0 \\ f_1 & g_2 & h_2 & 0 & 0 \\ 0 & f_2 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & g_{m-1} & h_{m-1} \\ 0 & 0 & 0 & f_{m-1} & g_m \end{bmatrix}$$

Note that each x_i influences at most nine elements of $X - GXG^*$.

Scalable Distributed Computations



The matrix $GXG^* - X$ is negative definite if and only if there exist y_i, z_i, w_i such that the following are negative definite:

$$W_1 = \begin{bmatrix} g_1 & h_1 \\ f_1 & g_2 \\ 0 & f_2 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \begin{bmatrix} g_1 & h_1 \\ f_1 & g_2 \\ 0 & f_2 \end{bmatrix}^* - \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 + w_1 & y_1 \\ 0 & y_1^* & z_1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} h_2 \\ g_3 \\ f_3 \end{bmatrix} x_3 \begin{bmatrix} h_2 \\ g_3 \\ f_3 \end{bmatrix}^* + \begin{bmatrix} w_1 & y_1 & 0 \\ y_1^* & z_1 - w_2 - x_3 & -y_2 \\ 0 & -y_2^* & -z_2 \end{bmatrix}$$

$$\vdots$$

$$W_{m-2} = \quad \dots$$

Conclusions

- Distributed Positive Test for Matrices
- Distributed Nonconservative System Verification
- Scalable Robustness Tests for Heterogeneous Systems